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COEXTENSIONS OF REGULAR SEMIGROUPS BY RECTANGULAR BANDS. I

BY

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ABSTRACT. This paper initiates a general study of the structure of a regular semigroup S via the maximum congruence ρ on S with the property that each ρ-class $e\rho$, for $e = e^2 \in S$, is a rectangular subband of S. Congruences of this type are studied and the maximum such congruence is characterized. A construction of all biordered sets which are coextensions of an arbitrary biordered set by rectangular biordered sets is provided and this is specialized to provide a construction of all solid biordered sets. These results are used to construct all regular idempotent-generated semigroups which are coextensions of a regular idempotent-generated semigroup by rectangular bands: a construction of normal coextensions of biordered sets is also provided.

1. Introduction. We say that a regular semigroup S is a coextension of a (necessarily regular) semigroup T by rectangular bands if there is a homomorphism ϕ from S onto T such that, for each idempotent $e \in S$, $e(\phi \circ \phi^{-1})$ is a rectangular subband of S. Thus any orthodox semigroup is a coextension of an inverse semigroup by rectangular bands since the kernel of the minimum inverse congruence on an orthodox semigroup consists of rectangular bands (see Schein **[28]**, Hall **[10]** and Meakin **[17]** for a description of this congruence and its kernel). This fact forms the basis for Hall's construction **[12]** of orthodox semigroups and for most subsequent work on the structure of orthodox semigroups.

Coextensions by rectangular bands have occurred in several other parts of the literature as well. For example D. Allen shows in **[1]** how to construct an arbitrary "max-principal" regular semigroup S in which $\mathcal{J} = \mathcal{D}$ by building a principally separated max-principal coextension of S by recangular bands: F. Pastijn **[25]** uses a somewhat similar construction of an arbitrary pseudo-inverse semigroup S by building a very nice coextension of S by rectangular bands. It is the aim of this paper and a later paper to describe how to construct all coextensions of regular semigroups by rectangular bands.

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We assume that the reader is familiar with the notions of a regular semigroup and an orthodox semigroup and the basic ideas and notation of semigroup theory–see Clifford and Preston [8] and Howie [14]. We denote the set of idempotents of a regular semigroup S by $E(S)$ and the set of inverses of an element a of S by $V(a)$. If T is a subset of S then $E(T)$ denotes the set of idempotents of S which are in T. The principal reference which we use in this paper is Nambooripad's work [20] (see also [19]): *we shall rely heavily on a familiarity with the notation and content of Nambooripad's paper and shall not repeat his definitions and results here*. In particular, the reader is assumed familiar with Nambooripad's notions of biordered sets, regular partial bands, sandwich sets, bimorphisms, τ-commutative E-squares, singular and degenerate E-squares, E-chains and his construction of regular partial bands and regular idempotent-generated semigroups. All biordered sets considered in this paper will be regular: consequently we shall omit the adjective "regular" and refer to a regular biordered set simply as a biordered set and a regular bimorphism as a bimorphism.

In addition, we need to recall some results of Nambooripad [21] on the *natural partial order* on a regular semigroup. This order is defined as follows. Let S be a regular semigroup and $x, y \in S$: define $x \leqslant y$ if and only if $R_x \leqslant R_y$ and $x = fy$ for some $f \in E(R_x)$ (equivalently, $x \leqslant y$ iff $L_x \leqslant L_y$ and $x = yf$ for some $f \in E(L_x)$). Nambooripad [21] showed that $\leqslant$ is a partial order on S whose restriction to $E(S)$ is ω. The order coincides with the usual natural partial order on S if S is inverse and is closely related to the structure of S. (The natural partial order on S may be used to express all products in S as products the trace of S–see [21].) The order is compatible with the multiplication in S if and only if S is pseudo-inverse, i.e. if and only if $|S(e, f)| = 1$ for each $e, f \in E(S)$ (see Nambooripad [21], [22]).

We shall need the following fact concerning the natural partial order.

THEOREM 1.1 (NAMBOORIPAD [21, THEOREM 1.8]). *Let ϕ be a homomorphism of the regular semigroup S onto the (regular) semigroup T. Then ϕ preserves and reflects the natural partial orders of S and T; i.e. if $x \leqslant y$ $(x, y \in S)$ then $x\phi \leqslant y\phi$; and if $w \leqslant z$ $(w, z \in T)$ and $y\phi = z$ for some $y \in S$ then there exists $x \in S$ such that $x \leqslant y$ and $x\phi = w$.*

2. The congruence ρ. Let S be a regular semigroup and let π (or $\pi(S)$) denote the relation on S defined by

$$\pi(S) = \{(x, y) \in S \times S\colon V(x) = V(y)\}.$$

It is well known that $\pi(S)$ plays an important role in determining the structure of an orthodox semigroup S. B. M. Schein [28] and independently T. E. Hall [10] showed that if S is an orthodox semigroup then $\pi(S)$ is a congruence on S and in fact $\pi(S)$ is the minimum inverse congruence on S, and Hall [11], [12] used this congruence to determine the structure of orthodox semigroups. D. B. McAlister and T. S. Blyth [16] studied orthodox semigroups which split over $\pi(S)$; several authors have studied congruences on orthodox semigroups via $\pi(S)$ (see for example the paper by Eberhart and Williams [9]). Yamada introduced $\pi(S)$ in [30]; $\pi(S)$ was studied from the point of view of its kernel by Meakin in [17] and

Nambooripad and Sitaraman studied congruences associated with $\pi(S)$ in **[23]**. In this section we initiate a study of the role of $\pi(S)$ in the structure of regular (not necessarily orthodox) semigroups.

An alternative characterization of $\pi(S)$ was given by Nambooripad in **[18]**. We have the following lemma.

LEMMA 2.1. *Let S be a regular semigroup. Then*

(a) $(x, y) \in \pi$ *if and only if for* $e \in E(R_y)$, $f \in E(L_y)$ *the mapping* λ_e: $E(R_x) \to E(R_y)$ *defined by* $\lambda_e(g) = eg$ *for* $g \in E(R_x)$ *is an* $\mathcal{L}$*-class preserving bijection of* $E(R_x)$ *onto* $E(R_y)$, *the mapping* ρ_f: $E(L_x) \to E(L_y)$ *defined by* $g\rho_f = gf$ *for* $g \in E(L_x)$ *is an* $\mathcal{R}$*-class preserving bijection of* $E(L_x)$ *onto* $E(L_y)$ *and* $y = exf$.

(b) *Suppose that* $V(x_1) = V(y_1)$ *and* $V(x_2) = V(y_2)$ *for some* $x_1, x_2, y_1, y_2 \in S$. *If* $L_{x_1} \cap R_{x_2}$ *and* $L_{y_1} \cap R_{y_2}$ *contain idempotents, then* $V(x_1x_2) = V(y_1y_2)$.

(c) *If* $V(x) = V(e)$ *and* $e \in E(S)$, *then* $x \in E(S)$.

PROOF. Part (a) was announced by Nambooripad in **[18]**. We provide a proof for completeness.

Suppose first that $V(x) = V(y)$, that $e \in E(R_y)$, $f \in E(L_y)$. There is an inverse y_1 of y (and of x) in $R_f \cap L_e$, and we have $exf = yy_1xy_1y = yy_1y = y$. Let $g \in E(R_x)$ and $h \in E(L_x)$. There is an inverse y_2 of x (and of y) in $R_h \cap L_g$, and hence there are idempotents $g_1 \in E(R_y) \cap L_{y_2}$ and $h_1 \in E(L_y) \cap R_{y_2}$. Then $g_1 = yy_2 = (exf)y_2 = ex(fy_2) = eg$ since fy_2 is easily seen to be an inverse of y (and hence of x) in R_{y_2}. Similarly $h_1 = hf$. Thus the mappings λ_e and ρ_f are mappings from $E(R_x)$ into $E(R_y)$ and from $E(L_x)$ into $E(L_y)$ respectively. Since $y_1 \in V(x)$ there are idempotents $e_1 \in E(R_x) \cap L_{y_1}$ and $f_1 \in E(L_x) \cap R_{y_1}$. By an argument dual to the above one sees that λ_{e_1} maps $E(R_y)$ into $E(R_x)$ and ρ_{f_1} maps $E(L_y)$ into $E(L_x)$: one easily checks that λ_{e_1} and λ_e [ρ_{f_1} and ρ_f] are mutually inverse mappings.

Suppose conversely that for all $e \in E(R_y)$ and $f \in E(L_y)$ we have $y = exf$ and the mappings λ_e and ρ_f satisfy the stated conditions. Let $x' \in V(x)$: then $xx' \in E(R_x)$ and $x'x \in E(L_x)$ and so $exx' \in E(R_y) \cap L_{x'}$ and $x'xf \in E(L_y) \cap R_{x'}$. It follows that there is an inverse y' of y in $H_{x'}$ and that $y'y = x'xf$, $yy' = exx'$ and that $x = x'(exx')$. Hence $y' = y'yy' = x'xfy' = x'exx'xfy' = x'exfy' = x'yy' = x'exx' = x'$ and so $x' \in V(y)$. Thus $V(x) \subseteq V(y)$ and dually $V(y) \subseteq V(x)$.

To prove part (b), note first that $x_1x_2 \in L_{x_2} \cap R_{x_1}$ and $y_1y_2 \in R_{y_1} \cap L_{y_2}$. Let $g \in E(R_{y_1y_2}) = E(R_{y_1})$. Since $(x_1, y_1) \in \pi$, the map λ_g: $E(R_{x_1}) = E(R_{x_1x_2}) \to E(R_{y_1}) = E(R_{y_1y_2})$ is an $\mathcal{L}$-class preserving bijection of $E(R_{x_1x_2})$ onto $E(R_{y_1y_2})$. Similarly, if $h \in E(L_{y_1y_2}) = E(L_{y_2})$, the map ρ_h is an $\mathcal{R}$-class preserving bijection of $E(L_{x_1x_2})$ onto $E(L_{y_1y_2})$. We need only show that $y_1y_2 = gx_1x_2h$.

Let $f = f^2 \in L_{y_1} \cap R_{y_2}$ and $e = e^2 \in L_{x_1} \cap R_{x_2}$. By part (a) we know that there are idempotents $f' \in L_{x_1} \cap R_{y_2} = L_e \cap R_f$ and $f'' \in R_{x_2} \cap L_{y_1} = R_e \cap L_f$ and that $f = f'f''$ and in addition $y_1 = gx_1f$ and $y_2 = fx_2h$. Hence $y_1y_2 = gx_1fx_2h = (gx_1f')(f''x_2h) = (gx_1)(x_2h) = gx_1x_2h$ and so the result follows from part (a).

To prove part (c), suppose that $V(x) = V(e)$ with $e \in E(S)$. Then $x \in V(e)$ and so $ex, xe \in E(S)$. Since $ex\mathcal{R}e$, $ex \in V(e) = V(x)$, so $x(ex) \in E(S) \cap H_x$. But $x(ex) = xex = x$, so $x \in E(S)$ as required.

The relation $\pi(S)$ is not necessarily a congruence on S: we denote by ρ (or $\rho(S)$) *the maximum congruence on S contained* in $\pi(S)$ and study regular semigroups via the congruence ρ. We first examine congruences contained in ρ.

For S, a regular semigroup, we denote by $\mathcal{V}(S)$ the equivalence relation

$$\mathcal{V}(S) = \{(x, y) \in S \times S\colon \mathcal{H}(V(x)) = \mathcal{H}(V(y))\}$$

where for $X \subseteq S$, $\mathcal{H}(X) = \{y \in S\colon y \in H_x, \text{ some } x \in X\} = \bigcup_{x \in X} H_x$.

PROPOSITION 2.2. *Let ν be a congruence on a regular semigroup S. Then the following are equivalent*:

(a) $\nu \subseteq \mathcal{V}(S)$;

(b) *for each $e \in E(S)$, $e\nu$ is a completely simple subsemigroup of S*;

(c) *if $x \leqslant y$ and $x\nu y$ for some $x, y \in S$, then $x = y$.*

PROOF. The equivalence of (b) and (c) was established by Nambooripad in **[21]**. Suppose that S satisfies (b) and let $(x, y) \in \nu$, $x' \in V(x)$. If $e = xx'$, $f = x'x$, then $yx' \,\nu\, e$ and so $H_{yx'}$ contains an idempotent e_1 such that $e_1 \,\nu\, e$. Let $f' \in E(L_y)$ and $h \in S(f', f)$. Then $yx' = yhx' \,\nu\, xhx' \,\omega\, e$. But $yx' \,\mathcal{L}\, hx' \,\mathcal{L}\, xhx' \,\nu\, e$, so $xhx' = e$ and so $e_1 \mathcal{L} e$. But since $yx' \in H_{e_1}$ it follows that $e_1 \omega^r e'$ for any $e' \in E(R_y)$ and so $e_1 y \mathcal{R} e_1$ and $e_1 y \leqslant y$. But $e_1 y \,\nu\, ey \,\nu\, ex = x$ and so $e_1 y = y$. Hence $R_y \cap L_e = R_y \cap L_{x'}$ contains the idempotent e_1. Dually $L_y \cap R_{x'}$ also contains an idempotent, so $H_{x'}$ contains an inverse of y. Similarly if $y' \in V(y)$, then $H_{y'}$ contains an inverse of x and so $\mathcal{H}(V(x)) = \mathcal{H}(V(y))$; i.e., $(x, y) \in \mathcal{V}(S)$. Hence $\nu \subseteq \mathcal{V}(S)$.

It is clear that (a) implies (b), so the equivalence of (a), (b) and (c) follows.

As a corollary we deduce the following proposition.

PROPOSITION 2.3. *The following are equivalent for a congruence ν on a regular semigroup S.*

(a) $\nu \subseteq \pi(S)$ (*equivalently*, $\nu \subseteq \rho(S)$);

(b) *for each $e \in E(S)$, $e\nu$ is a rectangular subband of S*;

(c) *ν satisfies the conditions*

(i) *$x \,\nu\, y$, $x \leqslant y$ implies $x = y$; and*

(ii) $\nu \cap \mathcal{H} = \iota_S$.

PROOF. Suppose first that ν satisfies (a). Since $\pi(S) \subseteq \mathcal{V}(S)$ it follows from Proposition 2.2 that each $e\nu$ ($e \in E(S)$) is a completely simple subsemigroup of S. Let $e \in E(S)$ and $e \,\nu\, x$ for some $x \in S$. Then $V(e) = V(x)$ so by Lemma 2.1(c), $x \in E(S)$. Thus for each $e \in E(S)$, $e\nu$ is a completely simple subband of S; i.e., $e\nu$ is a rectangular band.

Suppose next that ν satisfies (b) and let $(x, y) \in \nu$. By Proposition 2.2 we have $(x, y) \in \mathcal{V}(S)$. Let $x' \in V(x)$ and $y' \in V(y) \cap H_{x'}$. Now $xy' \,\nu\, yy'$, so $xy' \in E(S)$, so $xy' = xx'$. Similarly $yx' = yy'$, $x'y = y'y$, $y'x = x'x$. Thus

$$\begin{aligned} y' &= y'yy' = x'yx' \\ &= x'xx'yx'xx' = x'xy'yy'xx' \\ &= x'xy'xx' = x'xx'xx' = x'. \end{aligned}$$

Thus $V(x) \subseteq V(y)$ and similarly $V(y) \subseteq V(x)$; so $(x, y) \in \pi(S)$, and $\nu \subseteq \pi(S)$. Thus conditions (a) and (b) are equivalent.

It is clear that $\nu \subseteq \pi(S)$ implies ν satisfies (c). Suppose that ν satisfies (c). Then by Proposition 2.2, each $e\nu$ ($e \in E(S)$) is a completely simple subsemigroup of S. Since $\nu \cap \mathcal{H} = \iota_S$ it follows that $\mathcal{H}$ is the identity relation on $e\nu$ ($e \in E(S)$); so $e\nu$ is a rectangular band. Hence ν satisfies (b). This completes the proof.

We say that a biordered set E is *rectangular* if for $e, f \in E$ there are elements g, $h \in E$ such that $[\begin{smallmatrix} e & g \\ h & f \end{smallmatrix}]$ is an E-square. It is an easy matter to see that E is rectangular if and only if E is the biordered set of a completely simple semigroup or equivalently if and only if $\omega = \iota_E$.

It is clear from Proposition 2.3 that if ν is a congruence on S for which $\nu \subseteq \pi(S)$, then $E(\nu)$ is a biorder congruence on the biordered set $E(S)$ whose congruence classes are rectangular biordered subsets of $E(S)$. We next establish some basic properties of biorder congruences of this type.

Let θ be a bimorphism from the biordered set E onto the biordered set F and suppose that the congruence classes of $\theta \circ \theta^{-1}$ are all rectangular. For each $\alpha \in F$ denote the rectangular biordered set $\alpha\theta^{-1}$ by $E_\alpha = I_\alpha \times \Lambda_\alpha$. Elements of E_α are pairs $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$: in E_α we have $(i, \lambda)\mathcal{R}(j, \mu)$ iff $i = j$, $(i, \lambda)\mathcal{L}(j, \mu)$ iff $\lambda = \mu$, $\omega^r = \mathcal{R}$, $\omega^l = \mathcal{L}$ and the basic products in E_α are the obvious products of $\mathcal{R}$- or $\mathcal{L}$-related elements.

Lemma 2.4. (a) *If $\alpha\mathcal{R}\beta$ in F, then $|I_\alpha| = |I_\beta|$ and $E_\alpha \cup E_\beta$ is a rectangular biordered subset of E.*

(b) *If $\alpha\mathcal{L}\beta$ in F, then $|\Lambda_\alpha| = |\Lambda_\beta|$ and $E_\alpha \cup E_\beta$ is a rectangular biordered subset of E.*

(c) *The bimorphism θ reflects $\mathcal{R}$ and $\mathcal{L}$: i.e. if $\alpha\mathcal{R}$ $[\mathcal{L}]$ β and $e \in E_\alpha$ then there is some $f \in E_\beta$ such that $e\mathcal{R}$ $[\mathcal{L}]$ f.*

Proof. (a) Choose $e \in E_\alpha$, $f \in E_\beta$ and let $h \in S(e, f)$. Then $h\theta \in S(e\theta, f\theta) = S(\alpha, \beta) = \{\alpha\}$ so $h \in E_\alpha$. Since $e, h \in E_\alpha$ and $h \in \omega^l(e)$ it follows from the fact that E_α is rectangular that $e\mathcal{L}h$. The product hf exists in the biordered set E since $h\omega^r f$, so $(hf)\theta = (h\theta)(f\theta) = \alpha\beta = \beta$, so $hf \in E_\beta$. Since $f, hf \in E_\beta$ and $hf\,\omega\, f$ it follows that $hf = f$ since E_β is rectangular. Since $hf\mathcal{R}h$, we have $f\mathcal{R}h$. Thus, in the biordered set E, $e\mathcal{L}h\mathcal{R}f$. Similarly if $k \in S(f, e)$ then $e\mathcal{R}k\mathcal{L}f$. Part (a) now follows immediately; part (b) is proved by the dual argument. Part (c) also follows from the proof of (a) (or (b)).

Remark. In view of this lemma, we shall identify I_α with I_β if $\alpha\mathcal{R}\beta$ and Λ_α with Λ_β if $\alpha\mathcal{L}\beta$.

Lemma 2.5. *For all $e, f \in E$, $S(e, f)\theta = S(e\theta, f\theta)$; if $e \in E_\alpha$, $f \in E_\beta$ and $\gamma \in S(\alpha, \beta)$ then $M(e, f) \cap E_\gamma \neq \square$ and $M(e, f) \cap E_\gamma \subseteq S(e, f)$.*

Proof. Let $t \in M(e, f) \cap E_\gamma$ ($M(e, f) \cap E_\gamma \neq \square$ by [**20**, Proposition 2.14]). Suppose that $t_1 \in S(e, f) \cap E_\delta$. Then $\delta \in S(\alpha, \beta)$ so by [**20**, Corollary 2.11] there are elements μ, $\nu \in S(\alpha, \beta)$ such that $(\begin{smallmatrix} \gamma & \mu \\ \nu & \delta \end{smallmatrix})$ is an E-square in F. Let $t_2 \in S(t_1, t)$. Then $t_2\theta \in S(t_1\theta, t\theta) = S(\delta, \gamma) = \{\mu\}$, so $t_2 \in E_\mu$. Also $t_2 \in \omega^r(t)$ and t, t_2 are in the biordered set $E_\gamma \cup E_\mu$ which is rectangular by Lemma 2.4, so $t_2 \mathcal{R} t$. Similarly $t_2 \mathcal{L} t_1$. Since $t \in M(e,f)$ and $t_1 \in S(e,f)$ it follows from Nambooripad [**20**, Proposition 2.10] that there is an E-square

$$\begin{bmatrix} t_4 & t \\ t_3 & t_5 \end{bmatrix}$$

such that $t_3 \omega t_1$, $et_3 = et_4 \mathcal{R} et_5 = et$, $t_4 f = tf \mathcal{L} t_3 f = t_5 f$. Since $t_3 \omega t_1$ and t_3 is connected to t_1 by the E-chain $t_3 \mathcal{L} t_4 \mathcal{R} t_2 \mathcal{L} t_1$ of length less than four, it follows from Byleen, Meakin and Pastijn [**2**, Theorem 1.2] that $t_1 = t_3$. Hence $et_1 \mathcal{R} et$ and $t_1 f \mathcal{L} tf$ and so $t \in S(e,f)$ since $t_1 \in S(e,f)$.

THEOREM 2.6. *Let S be a regular semigroup, $e, f \in E(S)$ and ν a congruence on S contained in $\pi(S)$.*

(a) *If $e_1 \in e\nu$, $f_1 \in f\nu$ and $p_0 \in M(e,f)$ then there are elements $q_0 \in M(e_1,f)$, $r_0 \in M(e_1,f_1)$, $s_0 \in M(e,f_1)$ such that $p_0 \,\nu\, q_0 \,\nu\, r_0 \,\nu\, s_0$ and*

$$\begin{bmatrix} p_0 & q_0 \\ s_0 & r_0 \end{bmatrix}$$

is a rectangular band: if $p_0 \in S(e,f)$ then $q_0 \in S(e_1,f)$, $r_0 \in S(e_1,f_1)$, $s_0 \in S(e,f_1)$.

(b) *If $e_1 \in e\nu$, $f_1 \in f\nu$, $p_0 \in S(e,f)$ and $r_1 \in S(e_1,f_1)$ then there are elements $q_2 \in S(e_1,f)$ and $s_3 \in S(e,f_1)$ such that*

$$\begin{bmatrix} p_0 & q_2 \\ s_3 & r_1 \end{bmatrix}$$

is a rectangular band; in particular $\bigcup\{S(e_1,f_1): e_1 \in e\nu, f_1 \in f\nu\}$ is a rectangular band.

PROOF. (a) Let $e_1 \in e\nu, f_1 \in f\nu$ and $p_0 \in M(e,f)$ and let $q_0 = p_0 e_1$. Since $q_0 \,\nu\, p_0 e = p_0$ we have $q_0 \in E(S)$ and clearly $q_0 \in \omega^l(e_1)$. Since $q_0 \omega^r p_0$ and $q_0 \,\nu\, p_0$ it follows from Proposition 2.2(c) that $q_0 \mathcal{R} p_0$. We thus have $q_0 \,\nu\, p_0$, $q_0 \mathcal{R} p_0$ and $q_0 \in M(e_1,f)$. If we let $r_0 = f_1 p_0 e_1$ and $s_0 = f_1 p_0$ we see in a similar way that $p_0 \,\nu\, q_0 \,\nu\, r_0 \,\nu\, s_0$, $r_0 \in M(e_1,f_1)$, $s_0 \in M(e,f_1)$ and that $p_0 \mathcal{R} q_0 \mathcal{L} r_0 \mathcal{R} s_0 \mathcal{L} p_0$. It follows easily that $\left[\begin{smallmatrix} p_0 & q_0 \\ s_0 & r_0 \end{smallmatrix}\right]$ is a rectangular band.

Suppose now that $p_0 \in S(e,f)$ and let $h \in S(e_1,f)$. Then $p_0 \in V(ef)$, $h \in V(e_1 f)$ and $ef \,\nu\, e_1 f$, so $V(ef) = V(e_1 f)$. Thus $p_0 \in V(e_1 f)$ and so $e_1 f p_0 = e_1 p_0 \in E(S) \cap L_{p_0} \cap R_{e_1 f}$. Since $p_0 \,\nu\, q_0 = p_0 e_1$, we have $V(p_0) = V(p_0 e_1)$ and so $(e_1 p_0)(p_0 e_1) = e_1 p_0 e_1 = e_1 q_0 \in E(S) \cap R_{e_1 h} \cap L_{q_0}$ by Lemma 2.1(a). Since $e_1 h \in E(S) \cap R_{e_1 f}$, we have $e_1 q_0 \mathcal{R} e_1 h$. Similarly $q_0 f \mathcal{L} hf$, so $q_0 \in S(e_1,f)$ since $h \in S(e_1,f)$. Similarly $r_0 \in S(e_1,f_1)$ and $s_0 \in S(e,f_1)$. This completes the proof of (a).

(b) Let $e_1 \in e\nu, f_1 \in f\nu, p_0 \in S(e,f)$ and $r_1 \in S(e_1,f_1)$. By part (a) we know that $f_1 p_0 e_1 \in S(e_1,f_1)$ and since (by Nambooripad [**20**]) $S(e_1,f_1)$ is a rectangular band, we know that $f p_0 e_1 r_1 \in S(e_1,f_1) \cap R_{f_1 p_0 e_1} \cap L_{r_1}$. Let $q_2 = (p_0 e_1)(f_1 p_0 e_1 r_1) = (p_0 e_1)(f_1 p_0 e_1) r_1 = p_0 e_1 r_1$. Since $p_0 e_1 \,\nu\, f_1 p_0 e_1$ (by part (a)) it follows that $V(p_0 e_1) = V(f_1 p_0 e_1)$ and so $q_2 \in E(S) \cap R_{p_0 e_1} \cap L_{r_1}$ by Lemma 2.1(a). A dual argument shows that $s_3 = r_1 f_1 p_0 \in E(S) \cap L_{p_0} \cap R_{r_1}$, so $p_0 r_1 \in R_{p_0} \cap L_{r_1}$ and $r_1 p_0 \in R_{r_1} \cap L_{p_0}$. But $q_2 = p_0 e_1 r_1 \, \nu \, p_0 e r_1 = p_0 r_1$, so $p_0 r_1 \in E(S)$ by Lemma 2.1(c). Hence $q_2 = p_0 r_1$ (since $q_2, p_0 r_1 \in E(S)$ and $q_2 \mathcal{H} p_0 r_1$). Similarly $s_3 = r_1 p_0$ and so

$$\begin{bmatrix} p_0 & q_2 \\ s_3 & r_1 \end{bmatrix}$$

is a rectangular band. Now by the result of part (a) (with e and e_1 interchanged and f and f_1 interchanged) it follows that $fr_1 \in L_{r_1} \cap S(e_1, f)$; also $p_0 e_1 \in R_{p_0} \cap S(e_1, f)$. Since $S(e_1, f)$ is a rectangular band and $q_2 \in R_{p_0 e_1} \cap L_{fr_1}$ it follows that $q_2 \in S(e_1, f)$. Similarly $s_3 \in S(e, f_1)$. This completes the proof of part (b) and hence the theorem.

Remark 2.7. The situation described in Theorem 2.6 may be pictured in Diagram 1 below. In this diagram $p_i \in S(e, f)$, $q_i \in S(e_1, f)$, $r_i \in S(e_1, f_1)$, $s_i \in S(e, f_1)$ and $p_i \, \nu \, q_i \, \nu \, r_i \, \nu \, s_i$ for $i = 0, 1, 2, 3$.

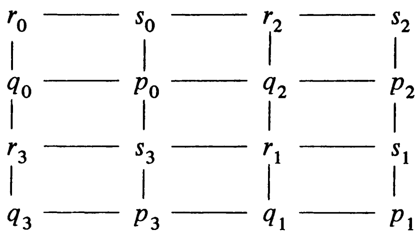

Diagram 1

We proceed now to a characterization of the congruence $\rho(S)$. We first introduce the following notion of similarity between principal left or right ideals.

Let S be a regular semigroup and $x, y \in S$. We say that the right ideals xS and yS are *similar*, written $xS \approx yS$, if there exist $e \in E(R_x)$ and $f \in E(R_y)$ such that $e \mathcal{L} f$ and $\theta_f = \lambda_f|_{E(xS)}$ is an $\mathcal{L}$-class preserving bijection of $\omega^r(e)$ onto $\omega^r(f)$ satisfying the following conditions:

(1) For all $g \in E(S)$ and $k \in S(g, e)$, $\theta_f(S(g, e)) = S(g, f)$ and $(ke, (\theta_f k) f) \in \pi(S)$.

(2) For all $u \in S$, $u' \in V(u)$ and $k \in S(u'u, e)$, $(uku', u\theta_f k u') \in \pi(S)$.

Similarity between principal left ideals is defined dually.

Proposition 2.8. *Let $e, f \in E(S)$ with $e \mathcal{L} f$. Then $\theta_f = \lambda_f|\omega^r(e)$ is an $\mathcal{L}$-class preserving bijection of $\omega^r(e)$ onto $\omega^r(f)$ satisfying conditions* (1) *and* (2) *of the foregoing definition if and only if $(e, f) \in \rho(S)$.*

Proof. Assume that θ_f satisfies conditions (1) and (2) of the definition. Then it is immediate from Lemma 2.1 that $V(e) = V(f)$. Let $u \in S$ and $u' \in V(u)$. If $k \in S(u'u, e)$ then by condition (1) of the foregoing definition we have $V(ke) = V(fkf)$ and $V(uku') = V(ufku')$. Now $uk \in L_u \cap R_{uku'}$ and $ufk \in L_{fk} \cap R_{ufku'} = L_k \cap R_{ufku'}$. Further

$$(ufku')(uk) = uf(ku'uk) = ufk$$

and $uku'\mathcal{L}ufku'$. Hence by Lemma 2.1(a), $V(uk) = V(ufk)$. Therefore by Lemma 2.1(b), $V(ke) = V((uk)(ke)) = V((ufk)(fkf)) = V(uf)$. Thus $(ue, uf) \in \pi$. If $h \in S(e, uu') = S(f, uu')$, then $eh \in \omega(e) \subseteq \omega^r(e)$ and $\theta_f(eh) = feh = fh$. Now $(eh)e = eh$, $\theta_f(eh)f = fhf = fh$ and so by condition (1) of the definition above $V(eh) = V(fh)$. Hence again by Lemma 2.1(b),

$$V(eu) = V((eh)(hu)) = V((fh)(hu)) = V(fu)$$

and so $(eu, fu) \in \pi$. Therefore $(e, f) \in \rho$.

Conversely assume that $(e, f) \in \rho$. Then for all $h \in \omega^r(e)$, $\theta_f h = fh \,\rho\, eh$ and so by Lemma 2.1, $fh \in \omega^r(f)$. Clearly $h\mathcal{L}fh$. Hence θ_f is an $\mathcal{L}$-class preserving mapping of $\omega^r(e)$ onto $\omega^r(f)$. To prove that θ_f is one-to-one, suppose that $\theta_f g = \theta_f h$ for $g, h \in \omega^r(e)$. Then $\theta_e\theta_f g = \theta_e\theta_f h$. But $\theta_e\theta_f g = e(fg) = g$ and so $g = h$.

Now let $g \in E(S)$ and $k \in S(g, e)$. Then $k \,\rho\, \theta_f k = fk$ and $ge \,\rho\, gf$. Hence $V(ge) = V(gf)$ and $V(k) = V(fk)$. Since $k \in V(ge)$ it follows that $gf \in V(k) = V(fk)$ and $fk \in \omega^l(g) \cap \omega^r(f)$. Hence $fk \in S(g, f)$ and so $\theta_f(S(g, e)) \subseteq S(g, f)$. Similarly $\theta_e(S(g, f)) \subseteq S(g, e)$ and so

$$S(g, f) = \theta_f(\theta_e(S(g, f))) \subseteq \theta_f(S(g, e)).$$

Also for any $k \in S(g, e)$, $ke \,\rho\, f(ke) \,\rho\, (fk)f$ and so $V(ke) = V(fkf)$. This proves that θ_f satisfies condition (1) of the foregoing definition. Condition (2) follows from the fact that, for all $k \in \omega^r(e)$, $k \,\rho\, \theta_f k$.

THEOREM 2.9. *Let S be a regular semigroup. Then $(x, y) \in \rho(S)$ iff $xS \approx yS$, $Sx \approx Sy$ and $y = exf$ for some $e \in E(R_y)$ and $f \in E(L_y)$.*

PROOF. Let $\sigma = \{(x, y): xS \approx yS, Sx \approx Sy$ and $y = exf$ for some $e \in E(R_y)$ and $f \in E(L_y)\}$. If $(x, y) \in \rho(S) = \rho$, then for any $x' \in V(x)$, $xx' \,\rho\, yx'$ and $x'x \,\rho\, x'y$. Since $xx'\mathcal{L}yx'$, by Proposition 2.8 we have $xS = xx'S \approx yx'S = yS$. Similarly $x'x\mathcal{R}x'y$ and by the dual of Proposition 2.8, $Sx \approx Sy$. Finally $y = (yx')x(x'y)$ and so $(x, y) \in \sigma$.

Conversely suppose that $(x, y) \in \sigma$. Then $xS \approx yS$ and so there exist $e_1 \in E(R_x)$, $f_1 \in E(R_y)$ such that $e_1\mathcal{L}f_1$ and $\theta_{f_1} = \lambda_{f_1}|\omega^r(e_1)$ is a bijection of $\omega^r(e_1)$ onto $\omega^r(f_1)$ satisfying the conditions of the foregoing definition. Then by Proposition 2.8, $(e_1, f_1) \in \rho$. Dually there exist $e_2 \in E(L_x)$ and $f_2 \in E(L_y)$ such that $e_2\mathcal{R}f_2$ and $(e_2, f_2) \in \rho$. Now $y = exf$ where $e \in E(R_y)$ and $f \in E(L_y)$. Since θ_{f_1} is an $\mathcal{L}$-class preserving bijection of $E(R_x)$ onto $E(R_y)$ there exists $e' \in E(R_x)$ such that $e = f_1e'$ and there exists $f' \in E(L_x)$ such that $f = f'f_2$. Hence $f_1xf_2 = (f_1e')x(f'f_2) = exf = y$ and so $V(x) = V(y)$. Therefore if x' is the inverse of x in $L_{e_1} \cap R_{e_2}$ then $e_1 = xx'$, $f_1 = yx'$, $e_2 = x'x$ and $f_2 = x'y$. Since $e_1 \,\rho\, f_1$, we have $x = e_1x \,\rho\, f_1x = yx'x$ and since $e_2 \,\rho\, f_2$, $yx'x = ye_2 \,\rho\, yf_2 = yx'y = y$. Hence $x \,\rho\, y$.

In what follows, for convenience we shall say that two $\mathcal{R}$-classes R, R' are *similar*, written $R \approx R'$, if for each $e \in E(R)$, λ_e is an $\mathcal{L}$-class preserving bijection from $E(R')$ onto $E(R)$. Similarity of two $\mathcal{L}$-classes is defined similarly. The following corollary is immediate.

COROLLARY 2.10. *The relation $\pi(S)$ on a regular semigroup S is a congruence if and only if S satisfies the following condition and its dual. If $x, y \in S$ then $R_x \approx R_y$ if and only if $xS \approx yS$.*

COROLLARY 2.11. *$\rho(S)$ is the identity congruence on a regular semigroup S if and only if no two distinct principal left or right ideals are similar.*

We say that S is *ρ-irreducible* if $\rho(S)$ is the identity congruence on S. The following proposition shows that every regular semigroup S has a ρ-irreducible homomorphic image.

PROPOSITION 2.12. *Let S be a regular semigroup and ν a congruence on S for which $\nu \subseteq \rho(S)$. Then $\rho(S/\nu) = \rho/\nu$. In particular, $\rho(S/\rho)$ is the identity congruence on S/ρ, i.e. S/ρ is ρ-irreducible.*

PROOF. We first prove that the congruence $\rho/\nu = \{(x\nu, y\nu) \in S/\nu \times S/\nu: (x, y) \in \rho\}$ satisfies conditions (i) and (ii) of Proposition 2.3(c). Note first that if $(x\nu, y\nu) \in \rho/\nu$ for some $x, y \in S$ then there exist elements x', $y' \in S$ such that $x'\nu = x\nu, y'\nu = y\nu$ and $x' \,\rho\, y'$. Now $(x', x) \in \nu \subseteq \rho$ and $(y', y) \in \nu \subseteq \rho$, so $x \,\rho\, y$.

Suppose that $x\nu \leqslant y\nu$ and $(x\nu, y\nu) \in \rho/\nu$. By Theorem 1.1 we can find $x' \in S$ such that $x'\nu = x\nu$ and $x' \leqslant y$. Thus $(x'\nu, y\nu) \in \rho/\nu$ and so $(x', y) \in \rho$, and so $x' = y$ by Proposition 2.3. Hence $x\nu = y\nu$ and ρ/ν satisfies condition (i) of Proposition 2.3(c).

Suppose now that $(x\nu, y\nu) \in \rho/\nu \cap \mathcal{H}$. Then $x \,\rho\, y$ and so $y = exf$ for some $e \in E(R_y)$, $f \in E(L_y)$ by Lemma 2.1. Thus $e\nu \mathcal{R} y\nu \mathcal{H} x\nu$, $f\nu \mathcal{L} y\nu \mathcal{H} x\nu$ and so $x\nu = (e\nu)(x\nu)(f\nu) = (exf)\nu = y\nu$, so $\rho/\nu \cap \mathcal{H} = \iota_{S/\nu}$ and ρ/ν satisfies condition (ii) of Proposition 2.3(c). Hence $\rho/\nu \subseteq \rho(S/\nu)$.

Suppose conversely that $\bar{\sigma}$ is a congruence on S/ν contained in $\rho(S/\nu)$ and let $\sigma = \{(x, y) \in S \times S: (x\nu, y\nu) \in \bar{\sigma}\}$. Then σ is a congruence on S such that $\nu \subseteq \sigma$. Let $(x, y) \in \sigma$ and $x \leqslant y$. Then $(x\nu, y\nu) \in \bar{\sigma}$ and $x\nu \leqslant y\nu$ so, by Proposition 2.3, $x\nu = y\nu$. Thus $x \leqslant y$ and $x\nu = y\nu$ and since $\nu \subseteq \rho$ it follows (again by Proposition 2.3) that $x = y$. Again, if $x \mathcal{H} y$ and $(x, y) \in \sigma$ then $x\nu \mathcal{H} y\nu$ and $(x\nu, y\nu) \in \bar{\sigma}$. Since $\bar{\sigma} \subseteq \rho(S/\nu)$ it follows that $x\nu = y\nu$. Hence $x \mathcal{H} y$ and $(x, y) \in \nu$ and since $\nu \subseteq \rho$, it follows that $x = y$. Thus $\sigma \subseteq \rho$ (by Proposition 2.3) and hence $\bar{\sigma} = \sigma/\nu \subseteq \rho/\nu$. In particular $\rho(S/\nu) \subseteq \rho/\nu$ and so $\rho(S/\nu) = \rho/\nu$ as required. The final statement in the proposition follows by taking $\nu = \rho$.

We next examine some connections between congruences contained in ρ and idempotent-separating congruences (congruences contained in $\mathcal{H}$).

PROPOSITION 2.13. *If S is a regular semigroup then*

(a) $\pi \cap \mathcal{H} = \iota_S$;

(b) $\pi \circ \mathcal{H} = \mathcal{H} \circ \pi \subseteq \mathcal{V}(S)$.

(c) *If σ and ν are congruences on S such that $\sigma \subseteq \mathcal{H}$ and $\nu \subseteq \pi$, then $\sigma \circ \nu = \nu \circ \sigma$; and in this case,*

(d) *$(\sigma \circ \nu)/\sigma$ is a congruence on S/σ contained in $\rho(S/\sigma)$.*

PROOF. Part (a) is obvious. To prove (b), let $(x, y) \in \pi \circ \mathcal{H}$, so that $(x, z) \in \pi$ and $(z, y) \in \mathcal{H}$ for some $z \in S$. Let $x' \in V(x) = V(z)$ and let $w = xx'yx'x$. Since $(x, z) \in \pi$, it follows from Lemma 2.1(a) that $L_{x'} \cap R_z$ contains an idempotent, so that $xx'y \in R_{xx'} \cap L_y$ and hence $w \in H_x$; i.e., $(x, w) \in \mathcal{H}$. Let $e \in E(R_y)$, $f \in E(L_y)$. Since $(x, z) \in \pi$ it follows from Lemma 2.1(a) that λ_e is an $\mathcal{L}$-class preserving bijection from $E(R_x) = E(R_w)$ onto $E(R_z) = E(R_y)$ and ρ_f is an $\mathcal{R}$-class preserving bijection from $E(L_w)$ onto $E(L_y)$. We need to show that $y = ewf$. Since $(x, z) \in \pi$ it follows from Lemma 2.1(a) that $exx' \in E(R_y)$ and $x'xf \in E(L_y)$, so $ewf = exx'yx'xf = y$ as required. Thus $(w, y) \in \pi$ and so $(x, y) \in \mathcal{H} \circ \pi$. Thus $\pi \circ \mathcal{H} \subseteq \mathcal{H} \circ \pi$ and it follows that $\pi \circ \mathcal{H} = \mathcal{H} \circ \pi$. Clearly $\pi \circ \mathcal{H} \subseteq \mathcal{V}(S)$ since $\mathcal{V}(S)$ is an equivalence relation containing π and $\mathcal{H}$.

To prove (c), suppose that σ and ν are congruences on S such that $\sigma \subseteq \mathcal{H}$ and $\nu \subseteq \pi$ and let $(x, y) \in \nu \circ \sigma$. Then we have $(x, z) \in \nu$ and $(z, y) \in \sigma$ for some $z \in S$ and again we let $x' \in V(x)$ and $w = xx'yx'x$. We have $w = xx'yx'x \ \sigma \ xx'zx'x = xx'x = x$ (since $x' \in V(x) = V(z)$) and $w = xx'yx'x \ \nu \ zx'yx'z = y$ since $zx' \in E(R_y)$ and $x'z \in E(L_y)$. Hence $(x, y) \in \sigma \circ \nu$.

To prove (d), let $\bar{S} = S/\sigma$ and $\bar{\nu} = (\nu \circ \sigma)/\sigma$; denote $x\sigma$ $(x \in S)$ by $\bar{x}$. It suffices to show that $\bar{\nu} \subset \bar{\pi} = \pi(\bar{S})$. Suppose that $(\bar{u}, \bar{v}) \in \bar{\nu}$. Now $\bar{\nu} = \{(\bar{x}, \bar{y}) \in S \times S: (x, y) \in \nu \circ \sigma = \sigma \circ \nu\}$ and $(x, y) \in \sigma \circ \nu$ implies $(x, w) \in \nu$ and $(w, y) \in \sigma$ for some $w \in S$. Hence we may assume that $(u, v) \in \nu$ and so $V(u) = V(v)$. In order to show that $V(\bar{u}) = V(\bar{v})$ it suffices to show that $V(\bar{u}) = \{\bar{u}': u' \in V(u)\}$. It is clear that if $u' \in V(u)$, then $\bar{u}' \in V(\bar{u})$; the converse follows from Lemma 1 of Hall [**13**]. Thus $(\bar{u}, \bar{v}) \in \bar{\pi}$ and so $\bar{\nu} \subseteq \bar{\pi}$ from which it follows that $\bar{\nu} \subseteq \rho(\bar{S}) = \rho(S/\sigma)$.

As a corollary, we have the following result.

COROLLARY 2.14. *If S is a ρ-irreducible regular semigroup then so is any $\mathcal{H}$-coextension of S.*

PROOF. Let T be a regular semigroup for which $T/\sigma \cong S$ for some idempotent-separating congruence σ on T. Then $\rho(T) \circ \sigma/\sigma \subseteq \rho(T/\sigma) = \rho(S) = \iota_S$, so $\rho(T) \circ \sigma/\sigma = \iota_S$. Let $(x, y) \in \rho(T)$ for some $x, y \in T$. Then $(x, y) \in \rho(T) \circ \sigma$ so $(x\sigma, y\sigma) \in \rho(T) \circ \sigma/\sigma$, so $x\sigma = y\sigma$. Hence $\rho(T) \subseteq \sigma$, so $\rho(T) \subseteq \rho(T) \cap \sigma \subseteq \pi(T) \cap \mathcal{H}$, so $\rho(T) = \iota_T$; i.e., T is ρ-irreducible. Thus any $\mathcal{H}$-coextension of S is ρ-irreducible.

A significant improvement on several of the results of this section may be obtained in the case in which S (or $S/\rho(S)$) is pseudo-inverse. The authors will provide a simpler characterization of the congruence $\rho(S)$ in this case as well as a characterization of ρ-irreducible pseudo-inverse semigroups in a later paper.

3. Coextensions of biordered sets by rectangular biordered sets. If S and S' are regular semigroups, we say that S is a *coextension of S' by rectangular bands* if there is a homomorphism $\theta: S \to S'$ from S onto S' such that the congruence classes of $\ker \theta$ which contain idempotents are rectangular bands. If E and E' are biordered sets then we say that E is a *coextension of E' by rectangular biordered sets* if there is a bimorphism $\theta: E \to E'$ from E onto E' such that, for each $e \in E$, $e(\theta \circ \theta^{-1})$ is a rectangular biordered subset of E: clearly if S is a coextension of S' by rectangular bands then $E(S)$ is a coextension of $E(S')$ by rectangular biordered sets. Any band may be viewed as a coextension of a semilattice by rectangular bands and any orthodox semigroup may be viewed as a coextension of an inverse semigroup by rectangular bands. The double four-spiral semigroup of Byleen, Meakin and

Pastijn [3] is a coextension of the fundamental four-spiral semigroup [2] by rectangular bands. Proposition 2.3 shows that if S is a regular semigroup and ν is a congruence on S contained in $\rho(S)$, then S is a coextension of S/ν by rectangular bands. In this section we show how to construct all coextensions of biordered sets by rectangular biordered sets. The results may be viewed as a generalization of Petrich's construction [26] of bands from semilattices and rectangular bands.

Let θ be a bimorphism from the biordered set E onto the biordered set F and suppose that, for each $e \in E$, $e(\theta \circ \theta^{-1})$ is a rectangular biordered set. For each $\alpha \in F$ denote the rectangular biordered set $\alpha\theta^{-1}$ by $E_\alpha = I_\alpha \times \Lambda_\alpha$ as in Lemmas 2.4 and 2.5.

If $\alpha \in F$ we denote $\{\gamma \in F: \gamma \mathcal{R} \alpha\}$ by $r(\alpha)$; similarly $l(\alpha) = \{\gamma \in F: \gamma \mathcal{L} \alpha\}$. Recall that $\kappa = \omega^r \cup \omega^l$. We introduce mappings $\phi^e_{r(\beta)}: I_\beta \to I_{\alpha\beta}$ and $\psi^e_{l(\beta)}: \Lambda_\beta \to \Lambda_{\beta\alpha}$, defined for certain elements $\beta \in F$ as follows.

Let $e \in E_\alpha$, $\beta \in F$ and $\beta \kappa \alpha$. Let $i \in I_\beta$, let $g = (i, \lambda) \in E_\beta$ and let $h \in S(e, g) \cap E_{\beta\alpha}$ (such an element h exists by Lemma 2.5). Since $\beta\alpha \mathcal{R} \beta$, $g \in E_\beta$, $h \in E_{\beta\alpha}$, and $h\omega^r g$, it follows from Lemma 2.4 that $h \mathcal{R} g$, so $h = (i, \mu)$, some $\mu \in \Lambda_{\beta\alpha}$. Now $h\omega^l e$ so $eh \in E$ and $eh \mathcal{L} h$, so $eh = (j, \mu)$ for some $j \in I_{\alpha(\beta\alpha)} = I_{\alpha\beta}$. We define a map $\phi^e_{r(\beta)}: I_\beta \to I_{\alpha\beta}$ by $\phi^e_{r(\beta)}(i) = j$. Note that if S is any regular semigroup for which $E(S) = E$ then $eh \mathcal{R} eg$ in S. Suppose now that $g' = (i, \lambda') \in E_{\beta'}$ for some $\beta' \in r(\beta)$ such that $\beta' \kappa \alpha$. Then in S, $eg \mathcal{R} eg'$, so the map $\phi^e_{r(\beta)} = \phi^e_{r(\beta')}$ is well defined. If $\gamma \in r(\beta)$ we define $\phi^e_{r(\gamma)} = \phi^e_{r(\beta)}$; thus a map $\phi^e_{r(\gamma)}$ is defined whenever $e \in E_\alpha$ and there is some $\beta \in r(\gamma)$ such that $\beta \kappa \alpha$. Let ϕ denote the set of all maps $\phi^e_{r(\gamma)}$ defined this way.

We define maps $\psi^e_{l(\gamma)}$ in a similar fashion if $e \in E_\alpha$ and there is some $\beta \in l(\gamma)$ such that $\beta \kappa \alpha$. Choose $g = (i, \lambda) \in E_\beta$, $f = (j, \lambda) \in S(g, e)$ and let $fe = (j, \mu)$ and define $\lambda\psi^e_{l(\gamma)} = \mu$. Denote the set of all such maps by ψ.

We write the maps $\phi^e_{r(\beta)}$ on the left and the maps $\psi^e_{l(\beta)}$ on the right. Conditions which these mappings must satisfy are provided in the following proposition. It is convenient to regard S as a regular semigroup with $E(S) = E$ and to think of the $\phi^e_{r(\beta)}$ and $\psi^e_{l(\beta)}$ mappings defined above as being induced by left and right translations by e respectively.

PROPOSITION 3.1. *The set $\phi \cup \psi$ of mappings defined above satisfies the following conditions and their duals.*

(P1) *If $\alpha(\mathcal{R} \cup \mathcal{L})\beta$ and $e = (i, \lambda) \in E_\alpha$ then $\phi^e_{r(\beta)}: I_\beta \to I_{\alpha\beta}$ is a constant mapping with value i.*

(P2) *If $e \in E_\alpha, f \in E_\beta, f\omega^r e$ and $\gamma\omega^r\beta$ then $\phi^e_{r(\gamma)}\phi^f_{r(\gamma)} = \phi^f_{r(\gamma)}$ and $\psi^e_{l(\gamma)}\psi^f_{l(\gamma\alpha)} = \psi^f_{l(\gamma)}$.*

(P3) *If $e \in E_\alpha$, $f \in E_\beta$, $\gamma \in S(\alpha, \beta)$, $\delta \in M(\alpha, \beta)$ and $h \in E_\gamma \cap M(e, f)$ then $\phi^e_{r(\delta)}\phi^f_{r(\delta)} = \phi^{eh}_{r(\mu)}\phi^{hf}_{r(\delta)}$ and $\psi^e_{l(\delta)}\psi^f_{l(\delta)} = \psi^{eh}_{l(\delta)}\psi^{hf}_{l(\nu)}$ where $\mu = (\gamma\beta)(\delta\beta)$ and $\nu = (\alpha\delta)(\alpha\gamma)$.*

REMARK. Since $\delta\beta\omega^l\gamma\beta$ it follows that $(\gamma\beta)(\delta\beta)\ \omega\ \gamma\beta$ and so $(\gamma\beta)(\delta\beta)\omega^r\gamma$, so $(\gamma\beta)(\delta\beta)\gamma\ \omega\ \gamma$ and hence $(\gamma\beta)(\delta\beta)\gamma\omega^l\alpha\gamma$. Thus the mapping $\phi^{eh}_{r(\mu)} = \phi^{eh}_{r((\gamma\beta)(\delta\beta)\gamma)}$ is defined. Similarly $\psi^{hf}_{l(\nu)}$ is defined. Furthermore, $\phi^{hf}_{r(\delta)}: I_{\delta\beta} \to I_{(\gamma\beta)(\delta\beta)}$, so $\phi^{eh}_{r(\mu)}\phi^{hf}_{r(\delta)}$ is defined; a similar argument shows that $\psi^{eh}_{l(\delta)}\psi^{hf}_{l(\nu)}$ is defined. The products eh and hf are of course basic products of E. The condition (P3) is self-dual.

PROOF OF PROPOSITION 3.1. (P1) Let $e = (i, \lambda) \in E_\alpha$ and $g = (j, \mu) \in E_\beta$ and $\alpha \mathcal{R} \beta$. Let $h \in S(e, g)$. Then $h \in E_\alpha$ and $h\omega^l e$, so $h \mathcal{L} e$. Thus $eh = e = (i, \lambda)$. By definition of $\phi^e_{r(\beta)}$ we have $\phi^e_{r(\beta)}(j) = i$. This proves (P1).

(P2) Let $f\omega^r e$, $f \in E_\beta$, $e \in E_\alpha$ and $\gamma\omega^r\beta$. Then we have $\gamma\omega^r\beta\omega^r\alpha$ in F. Let $g = (i, \lambda) \in E_\gamma$, let $h \in S(f, g) \cap E_{\gamma\beta}$, $h' \in S(e, fh)$ and let S be any regular semigroup with $E(S) = E$. In S we have $fh \mathcal{R} fg$ and $eh' \mathcal{R} e(fh) = fh$. Thus, if $fh = (j, \mu)$, then $eh' = (j, \nu)$ for some ν. We have $\phi^f_{r(\gamma)}(i) = j$; also $j \in I_\gamma = I_{\gamma\beta}$ since $h \in E_{\gamma\beta}$ and $\gamma\beta\ \omega\ \beta$ imply that $fh \in E_{\gamma\beta}$. Thus the mapping $\phi^e_{r(\gamma)}$ is defined and $\phi^e_{r(\gamma)}(j) = j$ (since $eh' \mathcal{R} fh$). It follows that $\phi^e_{r(\gamma)}\phi^f_{r(\gamma)}(i) = \phi^f_{r(\gamma)}(i)$ for all $i \in I_\gamma$, as required. The proof of the remaining statement in (P2) is proved in a similar fashion, by considering the associativity $(ge)f = g(ef) = gf$ in S and considering these products in terms of ψ-mappings.

(P3) Let $e \in E_\alpha$, $f \in E_\beta$, $\gamma \in S(\alpha, \beta)$, $\delta \in M(\alpha, \beta)$ and $h \in E_\gamma \cap M(e, f)$. By Lemma 2.5, $h \in S(e, f)$. Let $g = (i, \lambda) \in E_\delta$. Let $t \in S(f, g) \cap E_{\delta\beta}$: then $ft \in E_{\delta\beta}$ since $\delta\beta\ \omega\ \beta$. Since $\delta\beta \mathcal{R} \delta$ it follows from Lemma 2.4 that there is an element $t_1 \in E_\delta \cap L_g \cap R_{ft}$. Let $t_2 \in S(e, t_1) \cap E_\delta$: then $et_2 \in E_{\alpha\delta}$. Let $et_2 = (j, \lambda')$: then by definition of the ϕ-maps, $\phi^e_{r(\delta)}\phi^f_{r(\delta)}(i) = j$. Furthermore, if S is any regular semigroup with $E(S) = E$, then $(j, \lambda') = et_2 \mathcal{R} et_1 \mathcal{R} e(ft) = (ef)t \mathcal{R} (ef)g$.

Now let $g_1 = (i, \lambda_1) \in E_{\delta\beta} \cap R_g$ (such an element g_1 exists by Lemma 2.4). Let $t_3 \in S(hf, g_1) \cap E_{\delta\beta}$: then $(hf)t_3 \in E_{(\gamma\beta)(\delta\beta)} = E_\mu$. By the remark after the statement of Proposition 3.1, $\mu\gamma\ \omega\ \gamma$ and $\mu\gamma \mathcal{R} \mu$. Let $t_4 \in E_{\mu\gamma} \cap R_{hft_3}$ and let $t_5 \in S(e, t_4) \cap E_{\mu\gamma}$. Then $et_5 \in E_{\alpha(\mu\gamma)}$. Let $et_5 = (k, \lambda'')$; then by definition of the ϕ-maps, $\phi^{eh}_{r(\mu)}\phi^{hf}_{r(\delta)}(i) = k$. Furthermore, if S is any regular semigroup with $E(S) = E$ we have

$$\begin{aligned}(k, \lambda'') &= et_5 \mathcal{R} et_4 \mathcal{R} e(hft_3) \mathcal{R} e(hfg_1) = (eh)(hf)g_1 \mathcal{R} (eh)(hf)g \\ &= (ef)g \quad \text{since } h \in S(e, f) \\ &\mathcal{R} (j, \lambda');\end{aligned}$$

so $k = j$. Hence $\phi^{eh}_{r(\mu)}\phi^{hf}_{r(\delta)}(i) = \phi^e_{r(\delta)}\phi^f_{r(\delta)}(i)$ for all $i \in I_\delta$, as required. The remaining statement in (P3) is proved by the dual argument.

We proceed now to show that coextensions of biordered sets by rectangular biordered sets may be obtained from mappings satisfying these conditions.

Let F be a biordered set. For each $\alpha \in F$ let $E_\alpha = I_\alpha \times \Lambda_\alpha$ be a rectangular biordered set such that $I_\alpha = I_\beta$ if $\alpha \mathcal{R} \beta$, $I_\alpha \cap I_\beta = \square$ otherwise, $\Lambda_\alpha = \Lambda_\beta$ if $\alpha \mathcal{L} \beta$ and $\Lambda_\alpha \cap \Lambda_\beta = \square$ otherwise. Suppose that there are sets ϕ, ψ of well-defined maps $\phi^e_{r(\beta)} \in \phi$ and $\psi^e_{l(\beta)} \in \psi$ such that $\phi^e_{r(\beta)}$: $I_\beta = I_\gamma \to I_{\alpha\gamma}$ is defined when $e \in E_\alpha$ and γ is any element of $r(\beta)$ such that $\gamma\ \kappa\ \alpha$, and $\psi^e_{l(\beta)}$: $\Lambda_\beta = \Lambda_\gamma \to \Lambda_{\gamma\alpha}$ is defined when $e \in E_\alpha$ and γ is any element of $l(\beta)$ such that $\gamma\ \kappa\ \alpha$.

LEMMA 3.2. *Assume that the ϕ and ψ mappings satisfy* (P1) *and its dual. Let $E = \bigcup_{\alpha \in F} E_\alpha$, $e = (i, \lambda) \in E_\alpha$, $f = (j, \mu) \in E_\beta$. Define $f\omega^r e$ iff $\phi^e_{r(\beta)}(j) = j$ and $f\omega^l e$ iff $\mu\psi^e_{l(\beta)} = \mu$. If $f\omega^r e$ define $ef = f = (j, \mu)$ and $fe = (j, \mu\psi^e_{l(\beta)})$. If $f\omega^l e$ define $fe = f = (j, \mu)$ and $ef = (\phi^e_{r(\beta)}(j), \mu)$. Define $f \mathcal{R} e$ iff $f\omega^r e$ and $e\omega^r f$ and $f \mathcal{L} e$ iff $f\omega^l e$ and $e\omega^l f$ and define $\kappa = \omega^r \cup \omega^l$. Then if $f\omega^r e$ $[f\omega^l e]$ we must have $\beta\omega^r\alpha$ $[\beta\omega^l\alpha]$: $f \mathcal{R} e$ iff $\alpha \mathcal{R} \beta$ and $i = j$; $f \mathcal{L} e$ iff $\alpha \mathcal{L} \beta$ and $\lambda = \mu$. the products ef, fe, defined when $f\ \kappa\ e$ or $e\ \kappa\ f$ are well defined.*

PROOF. Suppose that $f\omega^r e$, so that $\phi^e_{r(\beta)}(j) = j$. In order that $\phi^e_{r(\beta)}$ is defined we must have some $\gamma \in r(\beta)$ such that $\gamma\ \kappa\ \alpha$. Since $\phi^e_{r(\beta)} = \phi^e_{r(\gamma)}: I_\gamma \to I_{\alpha\gamma}$ and $\phi^e_{r(\beta)}(j) = j$, we have $\gamma \mathcal{R} \alpha\gamma$; since $\gamma \mathcal{L} \alpha\gamma$ this forces $\gamma = \alpha\gamma$, so $\gamma\omega^r\alpha$, so $\beta\omega^r\alpha$. Suppose now that $f \mathcal{R} e$, so that $f\omega^r e$ and $e\omega^r f$. Then $\alpha \mathcal{R} \beta$ and $\phi^e_{r(\beta)}(j) = j$, $\phi^f_{r(\alpha)}(i) = i$. By (P1), $\phi^e_{r(\beta)}(j) = i$, so $i = j$. Thus $e \mathcal{R} f$ implies $i = j$. Conversely, if $i = j$, then by (P1), $\phi^e_{r(\beta)}(j) = i = j$ and $\phi^f_{r(\alpha)}(i) = j = i$ so $e\omega^r f$ and $f\omega^r e$, i.e., $e \mathcal{R} f$. To check that the products defined above are well defined we suppose $e \mathcal{R} f$; then $f\omega^r e$ so $ef = f = (j, \mu)$: also $e\omega^r f$ so $ef = (i, \lambda\psi^f_{l(\alpha)}) = (j, \lambda\psi^f_{l(\alpha)}) = (j, \mu)$ by the dual of (P1), i.e., the products ef defined from $e\omega^r f$ and $f\omega^r e$ coincide. Similarly the product fe is well defined if $e \mathcal{R} f$; a dual argument deals with the case $e \mathcal{L} f$. The product ef is clearly well defined if $e(\kappa \cup \kappa^{-1})f$ and $(e, f) \notin \mathcal{R} \cup \mathcal{L}$.

We have defined relations ω^r and ω^l and basic products ef for $e(\kappa \cup \kappa^{-1})f$. We consider E to be a partial algebra with domain $D_E = \kappa \cup \kappa^{-1}$. The sets $M(e, f)$ and $S(e, f)$ (for $e, f \in E$) are defined in the usual way:

$$M(e,f) = \{g \in E: g \in \omega^l(e) \cap \omega^r(f)\}, \quad \text{and}$$

$$S(e,f) = \{h \in M(e,f): eg\omega^r eh \text{ and } gf\omega^l hf, \forall g \in M(e,f)\}.$$

Assume now that the ϕ and ψ mappings satisfy the conditions (P2) (and its dual) and (P3) (where the products which occur in (P3) are basic products defined above). We show in a sequence of lemmas that E is a biordered set.

LEMMA 3.3. *The relations ω^r and ω^l on E are quasi-orders.*

PROOF. Let $e = (i, \lambda) \in E_\alpha$. Then $\phi^e_{r(\alpha)}(i) = i$ by (P1), so $e\omega^r e$; dually $e\omega^l e$, i.e., ω^r and ω^l are reflexive. Suppose that $g = (i, \lambda) \in E_\gamma$, $f \in E_\beta$, $e \in E_\alpha$ and $g\omega^r f$, $f\omega^r e$. Then by Lemma 3.2, $\gamma\omega^r\beta\omega^r\alpha$, and so

$$\begin{aligned}\phi^e_{r(\gamma)}(i) &= \phi^e_{r(\gamma)}\phi^f_{r(\gamma)}(i) \quad \left(\text{since } \phi^f_{r(\gamma)}(i) = i\right)\\ &= \phi^f_{r(\gamma)}(i) = i \quad \text{by (P2).}\end{aligned}$$

Hence $g\omega^r e$. Thus ω^r is transitive and similarly ω^l is transitive.

LEMMA 3.4. *E satisfies conditions* (B21), (B22), (B31) *and* (B32) *and their duals.*

PROOF. (B21). Suppose that $f = (i, \lambda) \in E_\beta$, $e \in E_\alpha$ and $f\omega^r e$. Then $\phi^e_{r(\beta)}(i) = i$ and $fe = (i, \lambda\psi^e_{l(\beta)}) \in E_{\beta\alpha}$ and $\beta\alpha \mathcal{R} \beta$, so by Lemma 3.4, $fe \mathcal{R} f$.

Also $\lambda\psi^e_{l(\beta)}\psi^e_{l(\beta\alpha)} = \lambda\psi^e_{l(\beta)}$ by (P2), so $fe\omega^l e$, and $\phi^e_{r(\beta\alpha)}(i) = \phi^e_{r(\beta)}(i) = i$, so $fe\omega^r e$. Hence $fe\ \omega\ e$, and (B21) is satisfied.

(B22). Suppose that $e \in E_\alpha$, $f = (i, \lambda) \in E_\beta$, $g = (i', \lambda') \in E_\gamma$ and that $g\omega^l f$ and $f, g \in \omega^r(e)$. Then $ge = (i', \lambda'\psi^e_{l(\gamma)}) \in E_{\gamma\alpha}$ and $fe = (i, \lambda\psi^e_{l(\beta)}) \in E_{\beta\alpha}$. Now $\beta\gamma\ \omega\ \beta \mathcal{R} \beta\alpha$, so $\beta\gamma\omega^r\beta\alpha\omega^r\alpha$ and $\beta\gamma \mathcal{L} \gamma$. Hence $(\beta\gamma)\alpha \mathcal{L} \gamma\alpha$ in F by axiom (B22). It follows that

$$\begin{aligned}\lambda'\psi^e_{l(\gamma)}\psi^{fe}_{l(\gamma\alpha)} &= \lambda'\psi^e_{l(\beta\gamma)}\psi^{fe}_{l((\beta\gamma)\alpha)} \quad (\text{since } \beta\gamma \mathcal{L} \gamma \text{ and } (\beta\gamma)\alpha \mathcal{L} \gamma\alpha)\\ &= \lambda'\psi^{fe}_{l(\beta\gamma)} \quad (\text{by (P2) since } fe\ \omega\ e \text{ by (B21)})\\ &= \lambda'\psi^{fe}_{l(\gamma)} = \lambda'\psi^f_{l(\gamma)}\psi^{fe}_{l(\gamma)} \quad \left(\text{since } g\omega^l f \text{ implies } \lambda' = \lambda'\psi^f_{l(\gamma)}\right)\\ &= \lambda'\psi^f_{l(\gamma)}\psi^e_{l(\gamma)} \quad \text{by P3,}\end{aligned}$$

since $\beta \in S(\beta, \alpha)$, $M(f, e) \cap E_\beta \neq \square$, $\gamma \in M(\beta, \alpha)$, $\beta\gamma\ \omega\ \beta$ and $\beta\gamma \mathcal{L} \gamma$. Hence $\lambda'\psi^{e}_{l(\gamma)}\psi^{fe}_{l(\gamma\alpha)} = \lambda'\psi^{e}_{l(\gamma)}$ (since $\lambda'\psi^{f}_{l(\gamma)} = \lambda'$), and so $ge\omega^l fe$.

(B31). Suppose that $g = (i, \lambda) \in E_\gamma$, $f \in E_\beta$ and $e \in E_\alpha$ and that $g\omega^r f\omega^r e$. Then $gf = (i, \lambda\psi^{f}_{l(\gamma)})$, $ge = (i, \lambda\psi^{e}_{l(\gamma)})$ and $(ge)f = (i, \lambda\psi^{e}_{l(\gamma)}\psi^{f}_{l(\gamma\alpha)})$. By (P2) we have $\psi^{e}_{l(\gamma)}\psi^{f}_{l(\gamma\alpha)} = \psi^{f}_{l(\gamma)}$, so $gf = (ge)f$, as required.

(B32). Suppose that $g = (i, \lambda) \in E_\gamma$, $f \in E_\beta$, $e \in E_\alpha$ and that $g\omega^l f$ and $f, g \in \omega^r(e)$. Then $fg = (\phi^{f}_{r(\gamma)}(i), \lambda) \in E_\gamma$, $(fg)e = (\phi^{f}_{r(\gamma)}(i), \lambda\psi^{e}_{l(\gamma)}) \in E_{\gamma\alpha}$, $ge = (i, \lambda\psi^{e}_{l(\gamma)}) \in E_{\gamma\alpha}$ and $(fe)(ge) = (\phi^{fe}_{r(\gamma\alpha)}(i), \lambda\psi^{e}_{l(\gamma)})$. Now $\beta \in S(\beta, \alpha)$, $f \in M(f, e) \cap E_\beta$, $\gamma \in M(\beta, \alpha)$ and $\beta\gamma \mathcal{R} (\beta\gamma)\alpha = (\beta\alpha)(\gamma\alpha)$ in F by axiom (B32), so by (P3), we have $\phi^{f}_{r(\gamma)}\phi^{e}_{r(\gamma)} = \phi^{f}_{r(\beta\gamma)}\phi^{fe}_{r(\gamma)}$. Since $\gamma\alpha\omega^l\beta\alpha$ the dual of (P2) implies that $\phi^{fe}_{r(\gamma\alpha)} = \phi^{fe}_{r((\beta\alpha)(\gamma\alpha))}\phi^{fe}_{r(\gamma\alpha)}$, so $\phi^{fe}_{r(\gamma)} = \phi^{fe}_{r(\beta\gamma)}\phi^{fe}_{r(\gamma)}$ since $\gamma \mathcal{R} \gamma\alpha$ and $\beta\gamma \mathcal{R} (\beta\alpha)(\gamma\alpha)$. Hence

$$\begin{aligned}
\phi^{f}_{r(\gamma)}\phi^{e}_{r(\gamma)} &= \phi^{f}_{r(\beta\gamma)}\phi^{fe}_{r(\gamma)} = \phi^{f}_{r(\beta\gamma)}\big(\phi^{fe}_{r(\beta\gamma)}\phi^{fe}_{r(\gamma)}\big)\\
&= \big(\phi^{f}_{r(\beta\gamma)}\phi^{fe}_{r(\beta\gamma)}\big)\phi^{fe}_{r(\gamma)}\\
&= \phi^{fe}_{r(\beta\gamma)}\phi^{fe}_{r(\gamma)} \quad \text{(by (P2) since } \beta\gamma\omega^r\beta \mathcal{R} \beta\alpha)\\
&= \phi^{fe}_{r(\gamma)},
\end{aligned}$$

so $\phi^{fe}_{r(\gamma)}(i) = \phi^{f}_{r(\gamma)}\phi^{e}_{r(\gamma)}(i) = \phi^{f}_{r(\gamma)}(i)$ since $g\omega^r e$ implies $\phi^{e}_{r(\gamma)}(i) = i$. Hence $(fg)e = (fe)(ge)$ as required. This completes the proof of the lemma, since the duals of (B21), (B22), (B31) and (B32) are verified in a dual fashion.

Lemma 3.5. *If $e \in E_\alpha$, $f \in E_\beta$, $\gamma \in S(\alpha, \beta)$ and $h \in M(e, f) \cap E_\gamma$, then $h \in S(e, f)$; furthermore, $S(e, f) \neq \square$, so E satisfies condition* (R).

Proof. Let $g = (i, \lambda) \in M(e, f) \cap E_\delta$. Then $\delta \in M(\alpha, \beta)$ so $\delta \prec \gamma$ in F, i.e., $\alpha\delta\omega^r\alpha\gamma$ and $\delta\beta\omega^l\gamma\beta$. Then $eg = (\phi^{e}_{r(\delta)}(i), \lambda) \in E_{\alpha\delta}$ and $\phi^{eh}_{r(\alpha\delta)}\phi^{e}_{r(\delta)}(i) = \phi^{eh}_{r(\alpha\delta)}\phi^{e}_{r(\delta)}\phi^{f}_{r(\delta)}(i)$ (since $g\omega^r f$) $= \phi^{eh}_{r(\alpha\delta)}\phi^{eh}_{r((\gamma\beta)(\delta\beta))}\phi^{hf}_{r(\delta)}(i)$ by (P3). Now in F, there exists an E-square

$$\begin{bmatrix} \delta, & \delta_1 \\ \delta_2, & \gamma' \end{bmatrix}$$

satisfying conditions (a), (b) and (c) of [**20**, Proposition 2.10]. By statement (b) we have $(\gamma\beta)(\delta\beta) = (\gamma\beta)(\delta_1\beta)$ and $\alpha\gamma' \mathcal{R} \alpha\delta$. By statement (c), $\gamma\delta_1 = \gamma'$ and by axiom (B32) and its dual we have $(\gamma\beta)(\delta\beta) = (\gamma\delta_1)\beta = \gamma'\beta \mathcal{R} \gamma'$ and $(\alpha\gamma)\gamma' = \alpha\gamma' \mathcal{R} \alpha\delta$. Thus we have

$$\begin{aligned}
\phi^{eh}_{r((\gamma\beta)(\delta\beta))} &= \phi^{eh}_{r(\gamma')} \quad \text{since } \gamma' \mathcal{R} (\gamma\beta)(\delta\beta)\\
&= \phi^{eh}_{r((\alpha\gamma)(\gamma'))}\phi^{eh}_{r(\gamma')} \quad \text{(by the dual of (P2))}\\
&= \phi^{eh}_{r(\alpha\delta)}\phi^{eh}_{r(\gamma')} = \phi^{eh}_{r(\alpha\delta)}\phi^{eh}_{r((\gamma\beta)(\delta\beta))},
\end{aligned}$$

and so $\phi^{eh}_{r(\alpha\delta)}\phi^{e}_{r(\delta)}(i) = \phi^{eh}_{r((\gamma\beta)(\delta\beta))}\phi^{hf}_{r(\delta)}(i) = \phi^{e}_{r(\delta)}\phi^{f}_{r(\delta)}(i)$ by (P3) $= \phi^{e}_{r(\delta)}(i)$ since $g\omega^r f$, and it follows that $eg\omega^r eh$. Similarly $gf\omega^l hf$, so $g \prec h$, and $h \in S(e, f)$, as required.

To verify that $S(e, f) \neq \square$, suppose that $e \in E_\alpha$, $f \in E_\beta$ and $t = (j, \mu) \in E_\gamma$ for some $\gamma \in S(\alpha, \beta)$. Then $(\phi^{f}_{r(\gamma)}(j), \mu\psi^{e}_{l(\gamma)}) \in E_\gamma \cap M(e, f)$, so $E_\gamma \cap M(e, f) \neq \square$. It follows from the first part of the proof that $S(e, f) \neq \square$, so E satisfies condition (R).

Lemma 3.6. *E is a biordered set.*

PROOF. It remains to verify that E satisfies condition (B4). Let $f \in E_\beta$, $g \in E_\gamma$ and $e \in E_\alpha$ and suppose that $f, g \in \omega^r(e)$. We need to check that $S(f, g)e = S(fe, ge)$.

Let $h \in S(f, g) \cap E_\delta$. Then $he \mathcal{R} h\omega^r g \mathcal{R} ge$, so $he\omega^r ge$, and $h, f \in \omega^r(e)$, $h\omega^l f$, so $he\omega^l fe$ by (B22). Thus $he \in M(fe, ge) \cap E_{\delta\alpha}$. Since F is a biordered set, F satisfies (B4), so $\delta\alpha \in S(\beta\alpha, \gamma\alpha)$, so by Lemma 3.5, $he \in S(fe, ge)$. Thus $S(f, g)e \subseteq S(fe, ge)$.

Conversely, let $t = (i, \lambda) \in S(fe, ge)$. Since F satisfies (B4), there is some $\delta \in S(\beta, \gamma)$ such that $t \in E_{\delta\alpha}$. Now by axioms (B32) and (B31), $\delta\alpha \mathcal{L} (\beta\alpha)(\delta\alpha) = (\beta\delta)\alpha \; \omega \; \beta\alpha \mathcal{R} \beta$ and $[(\beta\alpha)(\delta\alpha)]\beta = [(\beta\delta)\alpha]\beta = (\beta\delta)\beta = \beta\delta \mathcal{L} \delta$. So, $\psi^f_{l(\delta\alpha)} = \psi^f_{l((\beta\delta)\alpha)}$: $\Lambda_{\delta\alpha} = \Lambda_{(\beta\delta)\alpha} \to \Lambda_{[(\beta\delta)\alpha]\beta} = \Lambda_\delta$ and hence $u = (i, \lambda\psi^f_{l(\delta\alpha)}) \in E_\delta$. Clearly $u \mathcal{R} t\omega^r g$. Also,

$$\psi^f_{l(\delta\alpha)}\psi^f_{l(\delta)} = \psi^f_{l((\beta\delta)\alpha)}\psi^f_{l[((\beta\delta)\alpha)\beta]} = \psi^f_{l((\beta\delta)\alpha)} \quad \text{(by (P2))}$$
$$= \psi^f_{l(\delta\alpha)}.$$

It follows that $u\omega^l f$, so $u \in M(f, g) \cap E_\delta$. Since $\delta \in S(\beta, \gamma)$ it follows from Lemma 3.5 that $u \in S(f, g)$. Finally,

$$\begin{aligned}
\psi^f_{l(\delta\alpha)}\psi^e_{l(\delta)} &= \psi^f_{l(\delta\alpha)}\psi^f_{l(\delta)}\psi^e_{l(\delta)} \quad \text{(from the above)} \\
&= \psi^f_{l(\delta\alpha)}\psi^f_{l(\delta)}\psi^{fe}_{l((\beta\delta)\beta)} \quad \text{(by (P3))} \\
&= \psi^f_{l(\delta\alpha)}\psi^f_{l(\beta\delta)(\beta\delta)\beta} \\
&= \psi^f_{l((\beta\delta)\alpha)}\psi^{fe}_{l(((\beta\delta)\alpha)\beta)} \quad \text{(from the above)} \\
&= \psi^{fe}_{l((\beta\delta)\alpha)} \quad \text{(by (P2))} \\
&= \psi^{fe}_{l(\delta\alpha)},
\end{aligned}$$

so $ue = (i, \lambda\psi^f_{l(\delta\alpha)}\psi^e_{l(\delta)}) = (i, \lambda\psi^{fe}_{l(\delta\alpha)}) = (i, \lambda)$ since $t\omega^l fe$, i.e., $ue = t$. It follows that $S(fe, ge) \subseteq S(f, g)e$, as required. This completes the proof of the proposition.

We denote the biordered set E built this way by $E = E(F, \{E_\alpha\}, \phi, \psi)$.

PROPOSITION 3.7. *The mapping θ: $E \to F$ defined by $e\theta = \alpha$ if $e \in E_\alpha$ is a bimorphism from the biordered set E onto the biordered set F.*

PROOF. If $e \in E_\alpha$ and $f \in E_\beta$ then the products ef and fe are defined if and only if $\alpha \; \kappa \; \beta$ or $\beta \; \kappa \; \alpha$. Thus $(e, f) \in D_E$ implies $((e\theta), (f\theta)) \in D_F$. It is also clear that $(ef)\theta = (e\theta)(f\theta)$ when $(e, f) \in D_E$.

Let $e \in E_\alpha$, $f \in E_\beta$ and $h \in S(e, f) \cap E_\gamma$. Since $h\omega^l e$ it follows by definition of ω^l that $\gamma\omega^l\alpha$; similarly $\gamma\omega^r\beta$, so $\gamma \in M(\alpha, \beta)$. If $\delta \in M(\alpha, \beta)$ and $g = (i, \lambda) \in E_\delta$, then as in the proof of Lemma 3.5, $g' = (\phi^f_{r(\delta)}(i), \lambda\psi^e_{l(\delta)}) \in E_\delta \cap M(e, f)$, so $g' \prec h$ in E, from which it follows that $\delta \prec \gamma$ in F. Thus $\gamma \in S(\alpha, \beta)$ and so $S(e, f)\theta \subseteq S(e\theta, f\theta)$. Thus θ is a bimorphism.

We summarize the results obtained in this section in the following theorem.

THEOREM 3.8. *Let F be a biordered set and for each $\alpha \in F$ let $E_\alpha = I_\alpha \times \Lambda_\alpha$ be a rectangular biordered set such that $I_\alpha = I_\beta$ if $\alpha \mathcal{R} \beta$, $I_\alpha \cap I_\beta = \square$ otherwise, $\Lambda_\alpha = \Lambda_\beta$ if $\alpha \mathcal{L} \beta$ and $\Lambda_\alpha \cap \Lambda_\beta = \square$ otherwise. Suppose that there are well-defined sets ϕ, ψ of* *maps* $\phi^e_{r(\beta)}$: $I_\beta = I_\gamma \to I_{\alpha\gamma}$ *defined for* $e \in E_\alpha$ *and* $\gamma \in r(\beta)$ *with* $\gamma \kappa \alpha$, *and* $\psi^e_{l(\beta)}$: $\Lambda_\beta = \Lambda_\gamma \to \Lambda_{\gamma\alpha}$ *defined for* $e \in E_\alpha$ *and* $\gamma \in r(\beta)$ *with* $\gamma \kappa \alpha$. *If the* ϕ *and* ψ *mappings satisfy* (P1)–(P3) *and their duals then the products on the set* $E = \bigcup_{\alpha \in F} E_\alpha$ *defined in Lemma* 3.2 *are well-defined products and relative to this partial binary operation* $E = E(F, \{E_\alpha\}, \phi, \psi)$ *is a biordered set and the map* θ: $E \to F$ *defined by* $e\theta = \alpha$ *if* $e \in E_\alpha$ *is a bimorphism from* E *onto* F, *i.e.*, E *is a coextension of* F *by rectangular biordered sets. Every coextension of* F *by rectangular biordered sets is obtained this way.*

We close this section by providing a somewhat simpler construction of all solid biordered sets. A solid biordered set is one in which $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ (see Clifford **[6]**). Solid biordered sets are the biordered sets of completely regular semigroups (Clifford **[6]**). Several authors have studied the structure of completely regular semigroups (see, for example, Lallement **[15]**, Petrich **[27]**, Clifford and Petrich **[7]** and Warne **[29]**) and a structure theorem for solid biordered sets may in principle be extracted from these papers.

From the results of Clifford **[6]** it is easy to see that a biordered set E is solid if and only if E is a coextension of a semilattice F by rectangular biordered sets E_α ($\alpha \in F$). If F is a semilattice then $r(\alpha) = \{\alpha\} = l(\alpha)$ for all $\alpha \in F$ and $\omega^r = \omega^l = \omega = \leqslant$, the semilattice order in F, and so the maps $\phi^e_{r(\beta)}$ and $\psi^e_{l(\beta)}$ may be replaced by maps ϕ^e_β: $I_\beta \to I_\beta$ and $\psi^e_\beta = \Lambda_\beta \to \Lambda_\beta$ for $e \in E_\alpha$ and $\beta < \alpha$. Thus the conditions (P1)–(P3) and their duals reduce to conditions (S1)–(S3) (and their duals) below.

(S1) If $e = (i, \lambda) \in E_\alpha$ then ϕ^e_α is a constant map with value i.

(S2) If $e \in E_\alpha, f \in E_\beta, f\omega^r e$ and $\gamma \leqslant \beta$, then $\phi^e_\gamma \phi^f_\gamma = \phi^f_\gamma$ and $\psi^e_\gamma \psi^f_\gamma = \psi^f_\gamma$.

(S3) If $e \in E_\alpha$, $f \in E_\beta$, $\gamma = \alpha\beta$, $\delta \leqslant \gamma$ and $h \in E_\gamma \cap M(e, f)$, then $\phi^e_\delta \phi^f_\delta = \phi^{eh}_\delta \phi^{hf}_\delta$ and $\psi^e_\delta \psi^f_\delta = \psi^{eh}_\delta \psi^{hf}_\delta$. (Here the basic products and ω^r and ω^l relations in $E = \bigcup_{\alpha \in F} E_\alpha$ are defined again as in Lemma 3.2.)

Theorem 3.8 may be restated, replacing the biordered set F by a semilattice and the conditions (P1)–(P3) by (S1)–(S3), to yield a construction of all solid biordered sets, as follows.

COROLLARY 3.9. *Let* F *be a semilattice and for each* $\alpha \in F$ *let* $E_\alpha = I_\alpha \times \Lambda_\alpha$ *be a rectangular biordered set such that* $I_\alpha \cap I_\beta = \square$ *and* $\Lambda_\alpha \cap \Lambda_\beta = \square$ *if* $\alpha \neq \beta$. *Suppose that for each* $\alpha, \beta \in F$ *with* $\beta \leqslant \alpha$ *and each* $e \in E_\alpha$, *there are maps* ϕ^e_β: $I_\beta \to I_\beta$ *and* $\psi^e_\beta \Lambda_\beta \to \Lambda_\beta$ *which satisfy the conditions* (S1)–(S3) *and their duals. Then the products on the set* $E = \bigcup_{\alpha \in F} E_\alpha$ *defined in Lemma* 3.2 *are well-defined products relative to which* E *is a solid biordered set* (*which is a coextension of* F *by rectangular biordered sets*). *Every solid biordered set is obtained this way.*

REMARK. In §4 we shall see that we need to impose an additional restriction on the ϕ and ψ mappings of Corollary 3.9 in order that E admits a band structure; the conditions (S1)–(S3), together with this additional condition, will be equivalent to the conditions imposed by Petrich **[26]** in his construction of bands. This shows clearly the distinction between the concepts of "solid biordered set" and "orthodox biordered set" (i.e., the biordered set of some orthodox semigroup).

4. Regular idempotent-generated semigroups and regular partial bands. Recall that a partial algebra B is called a *regular partial band* if B is the partial algebra of idempotents of some regular semigroup S (relative to the multiplication in S). If B is a regular partial band then, relative to the basic products in B, B becomes a biordered set called the *biordered set determined by* B and denoted in this section by $E(B)$. If E is a biordered set then there may be many regular partial bands B and many regular idempotent-generated regular semigroups S for which $E(B) = E(S) = E$. We assume in this section that the reader is completely familiar with Nambooripad's construction **[20]** of all regular idempotent-generated semigroups determining a given biordered set E and with the construction of Clifford **[4]**, **[5]** and Nambooripad **[20]** of all regular partial bands determining E. All notation and terminology relevant to these constructions is also assumed here without further comment or explanation.

In **[24]** Pastijn introduced a generalization of the concept of the sandwich sets in a biordered set. We summarize his ideas below.

Let E be a biordered set and $e_1, \ldots, e_k \in E$. We define sets $M(e_1, \ldots, e_k)$ and $S(e_1, \ldots, e_k)$ as follows: $M(e_1, \ldots, e_k)$ is the set of all $(f_1, \ldots, f_{k-1}) \in E \times \cdots \times E$ such that

(i) $f_i \in M(e_i, e_{i+1})$ for $i = 1, \ldots, k-1$ and

(ii) $f_i e_{i+1} = e_{i+1} f_{i+1}$ for $i = 1, \ldots, k-2$.

Note that $e_1 f_1 \in \omega(e_1)$, $f_{k-1} e_k \in \omega(e_k)$ and $f_i e_{i+1} = e_{i+1} f_{i+1} \in \omega(e_{i+1})$ for $i = 1, \ldots, k-2$. Moreover

$$e_1 f_1 \mathcal{L} f_1 \mathcal{R} f_1 e_2 = e_2 f_2 \mathcal{L} \ \ldots \ \mathcal{L} f_i \mathcal{R} f_i e_{i+1} = e_{i+1} f_{i+1} \mathcal{L} f_{i+1} \mathcal{R} \ \ldots \ \mathcal{L} f_{k-1} \mathcal{R} f_{k-1} e_k.$$

We define sandwich sets $S(e_1, \ldots, e_k)$ as follows: $S(e_1, \ldots, e_k) = \{(f_1, \ldots, f_{k-1}) \in M(e_1, \ldots, e_k)$: $e_1 g_1 \omega^r e_1 f_1$ and $g_{k-1} e_k \omega^l f_{k-1} e_k$ for all $(g_1, \ldots, g_{k-1}) \in M(e_1, \ldots, e_k)\}$.

If $k = 2$, this reduces to the definition of the sandwich set of two elements of E.

The generalized sandwich sets are useful in locating products of n idempotents as the following theorem, due to Pastijn **[24]**, shows.

THEOREM 4.1 (PASTIJN [**24**, THEOREM 3.9]). *Let S be a regular semigroup and $e_1, \ldots, e_k \in E(S)$. Then $S(e_1, \ldots, e_k) \neq \square$ and if $(f_1, \ldots, f_{k-1}) \in S(e_1, \ldots, e_k)$ we have $e_1 \ldots e_k = e_1 f_1 e_2 f_2 \ldots e_{k-1} f_{k-1} e_k = w_S(C)$ where C is the E-chain $C = C(e_1 f_1, f_1, f_1 e_2, \ldots, f_i, f_i e_{i+1} = e_{i+1} f_{i+1}, \ldots, f_{k-1} e_k)$.*

The following proposition follows immediately from the definition of the sandwich sets.

PROPOSITION 4.2. *Let θ be a bimorphism from E to E' and let $e_1, \ldots, e_k \in E$. Then $S(e_1, \ldots, e_k)\theta \subseteq S(e_1\theta, \ldots, e_k\theta)$.*

Let E be a biordered set and S, S' regular idempotent-generated semigroups with $E(S) = E(S') = E$. We say that S *is richer than* S' (or S' *is poorer than* S) if ι_E extends to a homomorphism from S' onto S. We now prove a proposition which provides information about when bimorphisms extend to homomorphisms between regular idempotent-generated semigroups.

THEOREM 4.3. *Let S and S' be regular idempotent-generated semigroups with $E(S) = E$ and $E(S') = E'$ and let θ: $E \to E'$ be a bimorphism from E to E'. Then we have the following.*

(1) *The natural extension of θ to a mapping from S to S' is a well-defined homomorphism from S to S' if and only if the image of every E-cycle $\gamma \in \Gamma_S$ under $\mathcal{G}(\theta)$ is an E-cycle in $\Gamma_{S'}$. In particular if S and S' are fundamental then θ: $S \to S'$ is a homomorphism.*

(2) *If θ is surjective there exists a poorest regular idempotent-generated semigroup S_1' with $E(S_1') = E'$ such that θ extends to a homomorphism from S to S_1'.*

(3) *There exists a richest regular idempotent-generated semigroup S_1 with $E(S_1) = E$ such that θ extends to a homomorphism from S_1 to S'.*

PROOF. (1) Suppose that θ extends to a homomorphism from S to S' and let $\gamma = C(e = e_0, \ldots, e_n = e) \in \Gamma_S$. Then $\mathcal{G}(\theta)(\gamma) = C(e_0\theta, \ldots, e_n\theta)$; since $e_0 \ldots e_n = e$ we have $e_0\theta \ldots e_n\theta = e\theta$ so $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$. Suppose conversely that $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$ for each $\gamma \in \Gamma_S$. Then by [20, Theorem 6.9(c)], there exists a unique homomorhism ϕ: $B_{\Gamma_S} \to S'$ such that $\chi_{\Gamma_S} \circ E(\phi) = \theta$. If ψ_S is the isomorphism of S onto B_{Γ_S} then $E(\psi_S) = \chi_{\Gamma_S}$ and so $E(\psi_S \circ \phi) = \theta$. Therefore $\psi_S \circ \phi$ is a homomorphism of S into S' that extends θ. If S and S' are fundamental then Γ_S consists of all τ-commutative E-cycles in E and $\Gamma_{S'}$ consists of all τ-commutative E-cycles in E'; it is clear that if $\gamma \in \Gamma_{S'}$ then $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$.

(2) Since the image under $\mathcal{G}(\theta)$ of every τ-commutative E-cycle in E is τ-commutative in E' it follows that every E-cycle in $\Gamma_S\theta$ is τ-commutative. Now the intersection of any family of proper, closed sets of E-cycles is again proper and closed, so there is a least proper, closed set Γ' of E'-cycles containing $\Gamma_S\theta$. By [20, Theorem 6.9] there is a regular idempotent-generated semigroup S_1' such that $E(S_1') = E'$ and $\Gamma_{S'} = \Gamma'$. By (1) of the present theorem, θ extends to a homomorphism from S to S_1'. If S'' is a regular idempotent-generated semigroup with $E(S'') = E'$ such that θ extends to a homomorphism from S to S'' then again by (1), $\Gamma_S\theta \subseteq \Gamma_{S''}$, so $\Gamma' \subseteq \Gamma_{S''}$. Hence ι_E extends to a homomorphism of S_1' into S'' and so S_1' is poorer than S''.

(3) Let $\Gamma_{\theta^{-1}}$ denote the set of all τ-commutative E-cycles γ in E such that $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$. It is clear that all singular (and degenerate) E-cycles belong to $\Gamma_{\theta^{-1}}$. Also $\gamma \in \Gamma_{\theta^{-1}}$ implies $\gamma^{-1} \in \Gamma_{\theta^{-1}}$. Furthermore, if $\gamma \in \Gamma_{\theta^{-1}}$ and $e \in \omega(e_\gamma)$ then since γ is τ-commutative, $e * \gamma$ is a τ-commutative E-cycle and $\mathcal{G}(\theta)(e * \gamma) = e\theta * \mathcal{G}(\theta)(\gamma)$. Since $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$ and $\Gamma_{S'}$ is proper it follows that $e\theta * \mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$ and so $e * \gamma \in \Gamma_{\theta^{-1}}$. Hence $\Gamma_{\theta^{-1}}$ is a proper set of E-cycles.

Now let $C, C' \in \mathcal{G}(E)$ and $C \to_{\Gamma_{\theta^{-1}}} C'$. Then there exist $C_1, C_2 \in \mathcal{G}(E)$ and $\gamma \in \Gamma_{\theta^{-1}}$ such that $C = C_1C_2$ and $C' = C_1\gamma C_2$. Therefore, $\mathcal{G}(\theta)(C) = \mathcal{G}(\theta)(C_1)\mathcal{G}(\theta)(C_2)$ and $\mathcal{G}(\theta)(C') = \mathcal{G}(\theta)(C_1)\mathcal{G}(\theta)(\gamma)\mathcal{G}(\theta)(C_2)$, so $\mathcal{G}(\theta)(C) \to_{\Gamma_{S'}} \mathcal{G}(\theta)(C')$. Hence, in general, if $C \sim_{\Gamma_{\theta^{-1}}} C'$ in $\mathcal{G}(E)$ then $\mathcal{G}(\theta)(C) \sim_{\Gamma_{S'}} \mathcal{G}(\theta)(C')$ in $\mathcal{G}(E')$. Thus if γ is an E-cycle such that $\gamma \sim_{\Gamma_{\theta^{-1}}} C(e_\gamma)$ then $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$ since $\Gamma_{S'}$ is closed. Since an elementary $\Gamma_{\theta^{-1}}$-transition takes a τ-commutative E-cycle to a τ-commutative E-cycle (each $\gamma \in \Gamma_{\theta^{-1}}$ is τ-commutative) it follows that γ is τ-commutative and hence $\gamma \in \Gamma_{\theta^{-1}}$. Thus $\Gamma_{\theta^{-1}}$ is closed.

Thus by [20, Theorem 6.9] there is a regular idempotent-generated semigroup S_1 such that $E(S_1) = E$ and $\Gamma_{S_1} = \Gamma_{\theta^{-1}}$. Then by (1) of the present theorem θ extends to a homomorphism from S_1 to S'. If S_2 is any regular idempotent-generated semigroup with $E(S_2) = E$ such that θ extends to a homomorphism from S_2 to S', then $\Gamma_{S_2}\theta \subseteq \Gamma_{S'}$ and so by the definition of S_1, $\Gamma_{S_2} \subseteq \Gamma_{S_1}$. Hence ι_E extends to a homomorphism of S_2 onto S_1 and so S_1 is richer than S_2. This completes the proof of the theorem.

If θ is a homomorphism from a regular idempotent-generated semigroup S to a regular idempotent-generated semigroup S' we say that θ is *strong* if for each $s \in S$, $s \in E(S)$ if and only if $s\theta \in E(S')$. We have the following theorem.

THEOREM 4.4. *Let S' be a regular idempotent-generated semigroup and θ: $E \to E(S')$ a surjective bimorphism such that the congruence classes of* ker θ *are rectangular biordered sets. Let S be the richest regular idempotent-generated semigroup with $E(S) = E$ such that θ extends to a homomorphism from S onto S'. Then θ is a strong homomorphism if and only if every E-cycle γ in E such that $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}$ is τ-commutative.*

PROOF. Let $\mathcal{C}_{\theta^{-1}} = \{\gamma | \gamma$ is an E-cycle such that $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}\}$. Suppose first that θ: $S \to S'$ is a strong homomorphism and that $\gamma = C(e = e_0, \ldots, e_n = e) \in \mathcal{C}_{\theta^{-1}}$. Then $\gamma' = \mathcal{G}(\theta)(\gamma) = C(e_0\theta, \ldots, e_n\theta) \in \Gamma_{S'}$ and so $(e_0e_1 \ldots e_n)\theta = e_0\theta \ldots e_n\theta = e_0\theta$, so $e_0e_1 \ldots e_n \in E(S)$. But if C is an E-chain then $w_S(C) \in R_{e_C} \cap L_{f_C}$ and so $e_0 \ldots e_n \in H_{e_0} = H_e$ and it follows that $e_0 \ldots e_n = e$. Thus $\gamma \in \Gamma_S$ and so in particular γ is τ-commutative.

Suppose conversely that every E-cycle in $\mathcal{C}_{\theta^{-1}}$ is τ-commutative. Then $\mathcal{C}_{\theta^{-1}}$ coincides with the set $\Gamma_{\theta^{-1}}$ defined in the proof of Theorem 4.3(3), and so by the proof of that theorem, $\Gamma_S = \{\gamma | \gamma$ is an E-cycle with $\mathcal{G}(\theta)(\gamma) \in \Gamma_{S'}\}$. Suppose now that for some $e_1, e_2, \ldots, e_n \in E$ we have $e_1\theta \ldots e_n\theta = \bar{e} \in E(S')$. Let $(f_1, \ldots, f_{n-1}) \in S(e_1, \ldots, e_n)$. Then by Theorem 4.1 and Proposition 4.2, $(f_1\theta, \ldots, f_{n-1}\theta) \in S(e_1\theta, \ldots, e_n\theta)$ and $e_1 \ldots e_n = w_S(C)$, $(e_1\theta) \ldots (e_n\theta) = w_{S'}(C')$ where $C = C(e_1f_1, f_1, f_1e_2, \ldots, f_{n-1}e_n)$ and $C' = \mathcal{G}(\theta)(C)$. Now $\bar{e} \in M((f_{n-1}e_n)\theta, (e_1f_1)\theta)$, so by [20, Proposition 2.14] there exists $e \in E$ such that $e \in M(f_{n-1}e_n, e_1f_1)$ and $e\theta = \bar{e}$. Then $(f_{n-1}e_n)e \;\omega\; f_{n-1}e_n$ and

$$\begin{aligned}((f_{n-1}e_n)e)\theta &= (f_{n-1}e_n)\theta e\theta = f_{n-1}\theta e_n\theta e\theta \\ &= f_{n-1}\theta e_n\theta = (f_{n-1}e_n)\theta.\end{aligned}$$

Since the congruence classes of ker θ are rectangular, we have that $(f_{n-1}e_n)e = f_{n-1}e_n$. Hence $f_{n-1}e_n \mathcal{L} e$. Similarly $e \mathcal{R} e_1f_1$. It follows that $C(e, e_1f_1)CC(f_{n-1}e_n, e)$ is an E-cycle whose image is the $E(S')$-cycle $C(e\theta, (e_1f_1)\theta)C'C((f_{n-1}e_n)\theta, e\theta)$, which is in $\Gamma_{S'}$ since $w_{S'}(C') = \bar{e}$. Hence $C(e, e_1f_1)CC(f_{n-1}e_n, e) \in \Gamma_S$. This implies that $e(e_1f_1)w_S(C)(f_{n-1}e_n)e = e$ and so $w_S(C) = e$. Hence $e_1 \ldots e_n = e$ and in particular $e_1 \ldots e_n \in E(S)$. Thus the homomorphism θ is strong, as required.

We turn now to the question of building regular idempotent-generated semigroups which are coextensions of regular idempotent-generated semigroups by rectangular bands. We need to impose an additional condition on the ϕ and ψ mappings of §3 in this case.

THEOREM 4.5. *Let S' be a regular idempotent-generated semigroup with $E(S') = F$ and let $E = E(F, \{E_\alpha\}, \phi, \psi)$ be the coextension of F by the rectangular biordered sets E_α ($\alpha \in F$) described in Theorem* 3.8. *Suppose that in addition to* (P1)–(P3) *and their duals, the ϕ and ψ mappings satisfy the following condition.*

(P4) *Let $C(e_0, e_1, \ldots, e_n)$ be an E-cycle and $f = (i, \lambda)\ \omega\ e_0$; let $\theta(e_i) = \alpha_i$ ($i = 0, 1, \ldots, n$), $\theta(f) = \beta$ and let $\beta * C(\alpha_0, \alpha_1, \ldots, \alpha_n) = C(\beta_0, \beta_1, \ldots, \beta_n)$. Then if $C(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Gamma_S$ we have*

$$\lambda\psi^{e_1}_{l(\beta_0)}\psi^{e_2}_{l(\beta_1)} \cdots \psi^{e_n}_{l(\beta_{n-1})} = \lambda,$$

and

$$\phi^{e_n}_{r(\beta_{n-1})} \cdots \phi^{e_2}_{r(\beta_1)}\phi^{e_1}_{r(\beta_0)}(i) = i.$$

If S is the richest regular idempotent-generated semigroup with $E(S) = E$ such that the bimorphism θ: $E \to F$ extends to a homomorphism from S onto S' then S is a coextension of S' by the rectangular bands E_α ($\alpha \in F$). Conversely, if S is a coextension of S' by rectangular bands then the ϕ and ψ mappings associated with the extension of $F = E(S')$ by rectangular biordered sets satisfy condition (P4).

PROOF. Suppose that E is the coextension of F by the rectangular biordered sets E_α described in Theorem 3.8 and let $C(e_0, e_1, \ldots, e_n)$ be an E-cycle in E such that $\theta(e_i) = \alpha_i$ ($i = 0, 1, \ldots, n$) and $C(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Gamma_{S'}$: let $f = (i, \lambda)\ \omega\ e_0 = e_n$ and let $\theta(f) = \beta$. Since $C(\alpha_0, \alpha_1, \ldots, \alpha_n) \in \Gamma_S$, $C(\alpha_0, \ldots, \alpha_n)$ is τ-commutative in F and so $\beta * C(\alpha_0, \alpha_1, \ldots, \alpha_n) = C(\beta_0, \beta_1, \ldots, \beta_n)$ is a τ-commutative E-cycle in F. Then $C(e_0, e_1, \ldots, e_n)$ is τ-commutative in E if and only if $f\tau(e_0, e_1)\tau(e_1, e_2) \ldots \tau(e_{n-1}, e_n) = f$ for all $f \in \omega(e_0)$. But $f = (i, \lambda) \in E_{\beta_0}$ so $f\tau(e_0, e_1) = (\phi^{e_1}_{r(\beta_0)}(i), \lambda\psi^{e_1}_{l(\beta_0)}) \in E_{\beta_1}$ and so $f\tau(e_0, e_1)\tau(e_1, e_2) = (\phi^{e_2}_{r(\beta_1)}\phi^{e_1}_{r(\beta_0)}(i), \lambda\psi^{e_1}_{l(\beta_0)}\psi^{e_2}_{l(\beta_1)}) \in E_{\beta_2}$. Continuing this process by induction,

$$f\tau(e_0, e_1) \ldots \tau(e_{n-1}, e_n) = \left(\phi^{e_n}_{r(\beta_{n-1})} \cdots \phi^{e_2}_{r(\beta_1)}\phi^{e_1}_{r(\beta_0)}(i), \lambda\psi^{e_1}_{l(\beta_0)}\psi^{e_2}_{l(\beta_1)} \cdots \psi^{e_n}_{l(\beta_{n-1})}\right),$$

so we see that the ϕ and ψ mappings satisfy (P4) if and only if every E-cycle C in E such that $\mathcal{G}(\theta)(C) \in \Gamma_{S'}$ is τ-commutative. Thus by Theorem 4.4, θ is a strong homomorphism if and only if the ϕ and ψ mappings satisfy (P4). Clearly, θ is a strong homomorphism if and only if S is a coextension of S' by rectangular bands.

We shall now obtain the analogous result for regular partial bands.

If B_1 and B_2 are two regular partial bands then B_1 is called an *enrichment* of B_2 if $E(B_1) = E(B_2) = E$ and if the identity mapping ι_E is a (partial algebra) homomorphism from B_2 onto B_1 (i.e., if ef exists in B_2 then this product exists in B_1 and the products coincide). In this case we say that B_1 is *richer* than B_2 or that B_2 is *poorer* than B_1. From the results of Clifford [5] we see that corresponding to each biordered set E there is a *richest* regular partial band $\overline{E}_R$ determining E and a poorest regular partial band E_0 determining E. The richest regular partial band $\overline{E}_R$ is the fundamental closure of E (see Clifford [4]): it has the property that every τ-commutative E-square in E is a rectangular band in $\overline{E}_R$. The poorest regular partial band is the 'free' regular partial band generated by E–it is the regular partial band of the universal (or free) idempotent-generated semigroup determined by E (see Clifford [5] or Pastijn [24] or Nambooripad [20]).

A mapping θ from the regular partial band B_1 to the regular partial band B_2 is called a (regular partial band) *homomorphism* if θ is a bimorphism from $E(B_1)$ to $E(B_2)$ which preserves the products in B_1, i.e.,

(a) if $(e, f) \in D_{B_1}$ then $(e\theta, f\theta) \in D_{B_2}$ and $(ef)\theta = (e\theta)(f\theta)$; and

(b) $S(e, f)\theta \subseteq S(e\theta, f\theta) \forall e, f \in B_1$.

We say that θ is a *strong homomorphism* (or that B_1 is a *strong coextension* of B_2) if θ is a homomorphism and $(e, f) \in D_{B_1}$ if and only if $(e\theta, f\theta) \in D_{B_2}$.

We now prove a proposition which establishes when a bimorphism from $E(B)$ to $E(B')$ is a (regular partial band) homomorphism from B to B'.

PROPOSITION 4.6. *Let B and B' be regular partial bands with $E(B) = E$ and $E(B') = E$. Suppose that θ: $E \to E'$ is a bimorphism. Then we have the following*:

(1) *θ is a homomorphism of B into B' if and only if the image of every 2×2 rectangular band in B under $\mathcal{G}(\theta)$ is a 2×2 rectangular band in B'. In particular, if B and B' are fundamental, then θ: $B \to B'$ is a homomorphism.*

(2) *If θ is surjective there exists a poorest regular partial band B_1' with $E(B_1') = E'$ such that θ: $B \to B_1'$ is a homomorphism.*

(3) *There exists a richest regular partial band B_1 with $E(B_1) = E$ such that θ: $B_1 \to B$ is a homomorphism.*

PROOF. (1) Let $\mathcal{Q}$ $[\mathcal{Q}']$ denote the closed and effective set of E-squares of B $[B']$. Then $\Gamma_{\mathcal{Q}}$ is proper and $B_{\Gamma_{\mathcal{Q}}} = U(B)$ is the universal idempotent generated semigroup of B (cf. [**20**, Theorem 6.12]). The given condition implies that $\mathcal{G}(\theta)(\gamma)$ is commutative in B' for every $\gamma \in \Gamma_{\mathcal{Q}}$. Therefore by [**20**, Theorem 6.9(c)], θ extends to a homomorphism of $U(B)$ into $U(B')$. Since $B(U(B)) = B$ and $B(U(B')) = B'$ it follows that θ is a homomorphism of B into B'.

(2) By Theorem 4.3(2), there is a poorest idempotent-generated semigroup S' such that θ extends to a homomorphism of $U(B)$ into S'. Then θ is a homomorphism of B into $B(S')$ and $B(S')$ is obviously the poorest regular partial band on E' with this property.

(3) By Theorem 4.3(3), there is a richest idempotent-generated semigroup S such that θ extends to a homomorphism of S into $U(B')$. Then θ is a homomorphism of $B(S)$ into $B(U(B')) = B'$ and $B(S)$ is the richest regular partial band with this property.

Note that if θ: $B \to B'$ is a strong coextension of regular partial bands then B is, in particular, the richest regular partial band on $E(B)$ such that θ is a homomorphism. From the foregoing theorem, we deduce the following analogue for regular partial bands of Theorem 4.4 which can be proved using minor modifications to the proof of Theorem 4.4.

PROPOSITION 4.7. *Let B' be a regular partial band and θ: $E \to E(B')$ a surjective bimorphism such that the congruence classes of* ker θ *are rectangular biordered sets. Then there exists a regular partial B band with $E(B) = E$ such that θ: $B \to B'$ is a strong homomorphism if and only if every E-square γ in E such that $\mathcal{G}(\theta)(\gamma)$ is a 2×2 regular band in B' is τ-commutative.*

Strong coextensions of regular partial bands by rectangular bands may now be constructed by the following theorem.

THEOREM 4.8. *Let B' be a regular partial band with $E(B') = F$ and let $E = E(F, \{E_\alpha\}, \phi, \psi)$ be the coextension of F by the rectangular biordered sets E_α ($\alpha \in F$) described in Theorem* 3.8. *Suppose that in addition to* (P1)–(P3) *and their duals, the ϕ and ψ mappings satisfy the following condition*:

(P5) *If*

$$\begin{bmatrix} e_0 & e_1 \\ e_3 & e_2 \end{bmatrix}$$

is an E-square in E, $\theta(e_i) = \alpha_i$ for $i = 0, 1, 2, 3$ and

$$\begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 \end{bmatrix}$$

is a rectangular band in F, and if $t = (i, \lambda) \in \omega(e_0)$ and $\theta(t) = \beta$ then

$$\phi^{e_3}_{r(\beta)}(i) = \phi^{e_2}_{r(\beta)}(i) \quad \text{and} \quad \lambda\psi^{e_1}_{l(\beta)} = \lambda\psi^{e_2}_{l(\beta)}.$$

If B is the richest regular partial band with $E(B) = E$ such that the bimorphism θ: $E \to F$ extends to a homomorphism from B onto B' then B is a strong coextension of B' by rectangular bands. Conversely if B is a strong coextension of B' by rectangular bands then the ϕ and ψ mappings associated with the coextension of $F = E(B')$ by rectangular biordered sets satisfy condition (P5).

PROOF. This is proved by making a minor modification to the proof of Theorem 4.5, using Proposition 4.7 instead of Theorem 4.4. It is easy to see that condition (P4) may be replaced by (P5) when we restrict attention to those E-cycles which are in fact 4-cycles.

REMARK. We close this section by relating Theorem 4.8 to Petrich's construction of bands [26]. Clearly a band (viewed as a regular partial band) is a strong coextension of a semilattice by rectangular bands. Thus by Corollary 3.9 a band may be constructed from a semilattice F, a family of rectangular bands $E_\alpha = I_\alpha \times \Lambda_\alpha$ ($\alpha \in F$) and a family of mappings ϕ^e_β and ψ^e_β for $e \in E_\alpha$ and $\beta < \alpha$ which satisfy (S1)–(S3) and (P5). It is routine to see that conditions (S2) and (P5) imply that if $e, f \in E_\alpha$ and $\beta \leqslant \alpha$, then $\phi^e_\beta\phi^f_\beta = \phi^{ef}_\beta$ and $\psi^e_\beta\psi^f_\beta = \psi^{ef}_\beta$, so the map $\Phi_{\alpha,\beta}$: $E_\alpha \to \mathfrak{T}^*(I_\beta) \times \mathfrak{T}(\Lambda_\beta)$ given by $e\Phi_{\alpha,\beta} = (\phi^e_\beta, \psi^e_\beta)$ for $e \in E_\alpha$ is a homomorphism. The ϕ and ψ mappings reduce to the mappings considered by Petrich [26] and the conditions (S1)–(S3) and (P5) (and their duals) may fairly routinely be shown to be equivalent to Petrich's conditions. As we remarked earlier, this points out the distinction between "solid biordered sets" and "orthodox biordered sets".

In a later paper we shall show how to obtain a theorem describing the structure of coextensions of all regular semigroups by rectangular bands: this theorem is analogous to Hall's "spined product" theorem [12] describing the structure of orthodox semigroups.

5. Normal coextensions. Let θ be a bimorphism from the biordered set E onto the biordered set F and for each $\alpha \in F$ let E_α be the congruence class $E_\alpha = \{e \in E: e\theta = \alpha\}$. Each E_α is a biordered subset of E. We say that the coextension E of F is *normal* if it satisfies the following:

(N) if $\beta \,\omega\, \alpha$ in F and $e \in E_\alpha$, then there is a unique $f \in E_\beta$ such that $f \,\omega\, e$.

Clearly if E is a normal coextension of F then ω is trivial on each of the biordered subsets E_α ($\alpha \in F$) and so it follows immediately that each E_α is a rectangular biordered set, and so E is a coextension of F by the rectangular biordered sets E_α ($\alpha \in F$).

Major simplifications in the preceding theory for constructing coextensions occur in the normal case. We denote the value of a constant map θ from a set S to a set T by $\langle \theta \rangle$ in the proof of the following proposition.

PROPOSITION 5.1. *Let $E = E(F, \{E_\alpha\}, \phi, \psi)$ be a coextension of F by rectangular biordered sets. Then E is a normal coextension of F if and only if the mappings $\phi^e_{r(\beta)} \in \phi$ and $\psi^e_{l(\beta)} \in \psi$ are all constant maps.*

PROOF. Suppose first that the coextension is normal and that the mapping $\phi^e_{r(\beta)}$ is defined. Let $e \in E_\alpha$, $f = (i, \lambda) \in E_\beta$ and $\gamma \in r(\beta)$ such that $\gamma \,\kappa\, \alpha$. Then $\phi^e_{r(\beta)} = \phi^e_{r(\gamma)}: I_\gamma \to I_{\alpha\gamma} = I_{\alpha\gamma\alpha}$ and $\alpha\gamma\alpha \,\omega\, \alpha$. Let $\phi^e_{r(\beta)}(i) = j \in I_{\alpha\gamma\alpha}$ and let $g = (s, \mu)$ be the unique element of $E_{\alpha\gamma\alpha}$ such that $g \,\omega\, e$. Then $\phi^e_{r(\alpha\gamma\alpha)}(j) = \phi^e_{r(\alpha\gamma\alpha)}\phi^e_{r(\beta)}(i) = \phi^e_{r(\alpha\gamma)}\phi^e_{r(\gamma)} = \phi^e_{r(\gamma)}(i) = j$ by the dual of (P2), and $\psi^e_{l(\alpha\gamma\alpha)}(\mu) = \mu$ since $g\omega^l e$, so $(j, \mu) \,\omega\, e$ and $(j, \mu) \in E_{\alpha\gamma\alpha}$. Hence $j = s$, by normality. Since f was an arbitrary element of E_β, we see that $\phi^e_{r(\beta)}$ is a constant map (with $\langle \phi^e_{r(\beta)} \rangle = s$). A dual argument shows that the ψ-mappings are constant.

Suppose conversely that all maps in $\phi \cup \psi$ are constant. Let $\beta \,\omega\, \alpha$ in F and $e \in E_\alpha$ and let $\langle \phi^e_{r(\beta)} \rangle = i$ and $\langle \psi^e_{l(\beta)} \rangle = \lambda$. Clearly $(i, \lambda) \,\omega\, e$ and $(i, \lambda) \in E_\beta$. Suppose that $(j, \mu) \in E_\beta$ and $(j, \mu) \,\omega\, e$. Then $(j, \mu)\omega^r e$, so $\phi^e_{r(\beta)}(j) = j$ and so $j = i$: similarly $\mu = \lambda$ and so $(j, \mu) = (i, \lambda)$. Thus the coextension is normal.

Normal coextensions may be constructed by means of the following theorem, which provides a major simplification of Theorem 3.8 in this case.

THEOREM 5.2. *Let F be a biordered set and for each $\alpha \in F$ let $E_\alpha = I_\alpha \times \Lambda_\alpha$ be a rectangular biordered set such that $I_\alpha = I_\beta$ if $\alpha \mathcal{R} \beta$, $I_\alpha \cap I_\beta = \square$ otherwise, $\Lambda_\alpha = \Lambda_\beta$ if $\alpha \mathcal{L} \beta$ and $\Lambda_\alpha \cap \Lambda_\beta = \square$ otherwise. Let $\chi = \{\chi_{\alpha,\beta}: E_\alpha \to E_\beta \mid \beta\omega\alpha\}$ be a family of mappings $\chi_{\alpha,\beta}: E_\alpha \to E_\beta$ defined iff $\beta \,\omega\, \alpha$ and assume that χ satisfies the following conditions and their duals.*

(N1) *$\chi_{\alpha,\alpha} = \iota_{E_\alpha}$ for all $\alpha \in F$.*

(N2) *Let $e \in E_\alpha$, $f \in E_\beta$, $\alpha \mathcal{R} \beta$, $\gamma \,\omega\, \alpha$, $\delta \,\omega\, \beta$ and $\gamma \mathcal{R} \delta$. Then $e\chi_{\alpha,\gamma} \mathcal{R} f\chi_{\beta,\delta}$ in $E_\gamma \cup E_\delta$ if $e \mathcal{R} f$ in $E_\alpha \cup E_\beta$.*

(N3) *If $\gamma \,\omega\, \beta \,\omega\, \alpha$ then $\chi_{\alpha,\beta}\chi_{\beta,\gamma} = \chi_{\alpha,\gamma}$.*

On $E = \bigcup_{\alpha \in F} E_\alpha$ define quasi-orders ω^r and ω^l as follows: Let $e \in E_\alpha$ and $f \in E_\beta$; define $f\omega^r e$ iff $\beta\omega^r\alpha$ and $e\chi_{\alpha,\beta\alpha} \mathcal{R} f$ in $E_\beta \cup E_{\beta\alpha}$, and $f\omega^l e$ iff $\beta\omega^l\alpha$ and $e\chi_{\alpha,\alpha\beta} \mathcal{L} f$ in $E_\beta \cup E_{\alpha\beta}$. For $f\omega^r e$ define $ef = f$ and $fe = e\chi_{\alpha,\beta\alpha}$, and for $f\omega^l e$ define

$ef = e\chi_{\alpha,\alpha\beta}$ and $fe = f$. Then relative to these products E becomes a biordered set which is a normal coextension of F by the rectangular biordered sets E_α ($\alpha \in F$). Every normal coextension of F is obtained this way.

PROOF. Suppose first that we start with a biordered set F and a family χ of mappings satisfying (N1)–(N3) and their duals. Let $e \in E_\alpha$, $\gamma \,\kappa\, \alpha$, $\beta \mathcal{R} \gamma$ and $i \in I_\beta = I_\gamma$. Define $\phi^e_{r(\beta)}: I_\beta = I_\gamma \to I_{\alpha\gamma} = I_{\alpha\gamma\alpha}$ by $\phi^e_{r(\beta)}(i) = j$ where (j, μ) is the element $(j, \mu) = e\chi_{\alpha,\alpha\gamma\alpha} \in E_{\alpha\gamma\alpha}$. If for some $\gamma' \in r(\beta)$ we also have $\gamma' \,\kappa\, \alpha$, then $\alpha\gamma'\alpha \mathcal{R} \alpha\gamma' \mathcal{R} \alpha\gamma \mathcal{R} \alpha\gamma\alpha$, so by (N2), $e\chi_{\alpha,\alpha\gamma\alpha} \mathcal{R} e\chi_{\alpha\alpha\gamma'\alpha}$: thus the maps $\phi^e_{r(\beta)}$ are well defined and are clearly constant maps. The maps $\psi^e_{l(\beta)}$ are defined dually: if $e \in E_\alpha$, $\gamma \mathcal{L} \beta$, $\gamma \,\kappa\, \alpha$ and $e\chi_{\alpha,\alpha\gamma\alpha} = (j, \mu)$ then for all $\lambda \in \Lambda_\beta = \Lambda_\gamma$ we define $\lambda\psi^e_{l(\beta)} = \mu$; again the $\psi^e_{l(\beta)}$ maps are well-defined constant maps.

If $\alpha(\mathcal{R} \cup \mathcal{L})\beta$, then $\alpha\beta\alpha = \alpha$, so $\chi_{\alpha,\alpha\beta\alpha} = \chi_{\alpha,\alpha}$. Hence, by (N1) the mappings $\phi^e_{r(\beta)}$, $\chi^e_{l(\beta)}$ satisfy condition (P1). Furthermore, by definition of the ϕ and ψ mappings it is clear that the relations ω^r and ω^l coincide with the corresponding relations defined in Lemma 3.2 and the basic products coincide with the basic products defined in Lemma 3.2, so (by Lemma 3.2) the relations ω^r and ω^l are quasi-orders and the basic products are well defined.

Suppose now that $e \in E_\alpha$, $f \in E_\beta$, $f\omega^r e$ and $\gamma\omega^r\beta$ and let $(i, \lambda) \in E_\gamma$. Let $\lambda\psi^e_{l(\gamma)} = \mu \in \Lambda_{\gamma\alpha}$, $\mu\psi^f_{l(\gamma\alpha)} = \nu$ and $\lambda\psi^f_{l(\gamma)} = \tau$. Then $e\chi_{\alpha,\alpha\gamma\alpha} = (j, \mu)$ (for some $j \in I_{\alpha\gamma\alpha}$), $f\chi_{\beta,\beta(\gamma\alpha)\beta} = (k, \nu)$ (for some $k \in I_{\beta(\gamma\alpha)\beta}$) and $f\chi_{\beta,\beta\gamma\beta} = (s, \tau)$ (for some $s \in I_{\beta\gamma\beta}$). Since $\beta(\gamma\alpha)\beta = \beta\gamma\beta$ in F it follows that $(k, \nu) = (s, \tau)$, so $\nu = \tau$ and so $\psi^e_{l(\gamma)}\psi^f_{l(\gamma\alpha)} = \psi^f_{l(\gamma)}$. Let $\phi^f_{r(\gamma)}(i) = u \in I_\gamma$ and $\phi^e_{r(\gamma)}(u) = v$. Then $f\chi_{\beta,\beta\gamma} = (u, \rho)$ and $e\chi_{\alpha,\gamma\alpha} = (v, \pi)$ for suitable ρ, π. Now $f \mathcal{R} fe = e\chi_{\alpha,\beta\alpha}$, $\gamma\beta \;\omega\; \beta$, $\gamma(\beta\alpha) \;\omega\; \beta\alpha$ and $\gamma\beta \mathcal{R} \gamma(\beta\alpha)$, so by (N2),

$$f\chi_{\beta,\gamma\beta} \mathcal{R} fe\chi_{\beta\alpha,\gamma(\beta\alpha)} = e\chi_{\alpha,\beta\alpha}\chi_{\beta\alpha,\gamma(\beta\alpha)} = e\chi_{\alpha,\gamma(\beta\alpha)} \quad \text{(by (N3))}$$
$$\mathcal{R} e\chi_{\alpha,\gamma\alpha} \quad \text{(by (N2) since } \gamma\alpha \mathcal{R} \gamma(\beta\alpha)).$$

Hence $u = v$ and so $\phi^e_{r(\gamma)}\phi^f_{r(\gamma)} = \phi^f_{r(\gamma)}$. Hence the ϕ and ψ mappings satisfy (P2).

Now suppose that $e \in E_\alpha$, $f \in E_\beta$, $\gamma \in S(\alpha, \beta)$, $\delta \in M(\alpha, \beta)$ and $h \in E_\gamma \cap M(e, f)$ and let $\lambda \in \Lambda_\delta$. Suppose that $\lambda\psi^e_{l(\delta)} = \mu \in \Lambda_\delta$ and that $\mu\psi^f_{l(\delta)} = \tau$, so that $f\chi_{\beta,\beta\delta\beta} = (j, \tau)$ for some $j \in I_{\beta\delta\beta}$. Now there exists an E-square

$$\begin{bmatrix} \delta & \delta_1 \\ \delta_2 & \gamma' \end{bmatrix}$$

in F satisfying conditions (a), (b) and (c) of [**20**, Proposition 2.10]. Thus we have

$$(\alpha\delta)(\alpha\gamma) = (\alpha\delta_2)(\alpha\gamma) = \alpha(\delta_2\gamma) = \alpha\gamma' \mathcal{L} \gamma'$$

and

$$(\gamma\beta)\gamma'(\gamma\beta) = \gamma'(\gamma\beta) = (\gamma'\gamma)\beta = \gamma'\beta \mathcal{L} \delta\beta.$$

Let $\lambda\psi^{eh}_{l(\delta)} = \mu_1 \in \Lambda_{(\alpha\delta)(\alpha\gamma)} = \Lambda_{\gamma'}$ and $\mu_1\psi^{hf}_{l((\alpha\delta)(\alpha\gamma))} = \mu_1\psi^{hf}_{l(\gamma')} = \tau_1$, so that $hf\chi_{\gamma\beta,(\gamma\beta)\gamma'(\gamma\beta)} = (j_1, \tau_1)$ for some j_1. Since $hf = f\chi_{\beta,\gamma\beta}$, we have

$$(j_1, \tau_1) = f\chi_{\beta,\gamma\beta}\chi_{\gamma\beta,\gamma'\beta} = f\chi_{\beta,\gamma'\beta} \quad \text{(by (N3))}$$
$$\mathcal{L} f\chi_{\beta,\delta\beta} \quad \text{(by (N2))}$$
$$= f\chi_{\beta,\beta\delta\beta} = (j, \tau)$$

so $\tau_1 = \tau$. It follows that $\psi^e_{l(\delta)}\psi^f_{l(\delta)} = \psi^{eh}_{l(\delta)}\psi^{hf}_{l((\alpha\delta)(\alpha\gamma))}$ and a dual argument shows that $\phi^e_{r(\delta)}\phi^f_{r(\delta)} = \phi^{eh}_{r((\gamma\beta)(\delta\beta))}\phi^{hf}_{r(\delta)}$, and the ϕ and ψ mappings satisfy (P3).

Thus by Theorem 3.8, $E = E(F, \{E_\alpha\}, \phi, \psi)$ is a coextension of F by the rectangular biordered sets E_α ($\alpha \in F$); the coextension is normal by Proposition 5.1 since the ϕ and ψ maps are constant maps.

Suppose conversely that $E = E(F, \{E_\alpha\}, \phi, \psi)$ is a normal coextension of F by the rectangular bands E_α ($\alpha \in F$). If $\beta\ \omega\ \alpha$ and $e \in E_\alpha$ define $e\chi_{\alpha,\beta} = f$ if f is the element in E_β with $f\ \omega\ e$. It is easy to check from the definition of normal coextension that the $\chi_{\alpha,\beta}$ mappings satisfy (N1) and (N3). To check (N2), suppose that $\alpha \mathcal{R} \beta$, $\gamma \mathcal{R} \delta$, $\gamma\ \omega\ \alpha$, $\delta\ \omega\ \beta$ and that $e \mathcal{R} f$ for some $e \in E_\alpha$, $f \in E_\beta$. If $e\chi_{\alpha,\gamma} = (i, \lambda)$ and $f\chi_{\beta,\delta} = (j, \mu)$ then $(i, \mu)\omega f$ and $(i, \mu) \in E_\delta$ so we must have $i = j$ by normality. This completes the proof of the theorem.

We denote the coextension obtained in this manner by $E(F, \{E_\alpha\}, \chi)$.

As a corollary we deduce the following result.

Corollary 5.3. *Let F be a biordered set and for each $\alpha \in F$ let $E_\alpha = I_\alpha \times \Lambda_\alpha$ be any rectangular biordered set such that $I_\alpha = I_\beta$ if $\alpha \mathcal{R} \beta$, $I_\alpha \cap I_\beta = \square$ otherwise, $\Lambda_\alpha = \Lambda_\beta$ if $\alpha \mathcal{L} \beta$ and $\Lambda_\alpha \cap \Lambda_\beta = \square$ otherwise. Then there is a family χ of mappings such that $E(F, \{E_\alpha\}, \chi)$ is a normal coextension of F by the rectangular biordered sets E_α ($\alpha \in F$).*

Proof. In each E_α ($\alpha \in F$) choose an element $\bar{\alpha}$ such that $\bar{\alpha} \mathcal{R} \bar{\beta}$ in $E_\alpha \cup E_\beta$ if $\alpha \mathcal{R} \beta$ and $\bar{\alpha} \mathcal{L} \bar{\beta}$ in $E_\alpha \cup E_\beta$ if $\alpha \mathcal{L} \beta$ (such a choice can always be made by the axiom of choice). For $\beta\ \omega\ \alpha$, $\beta \neq \alpha$, define $\chi_{\alpha,\beta}\colon E_\alpha \to E_\beta$ by $e\chi_{\alpha,\beta} = \bar{\beta}$ for all $e \in E_\alpha$. Define $\chi_{\alpha,\alpha} = \iota_{E_\alpha}$ for all $\alpha \in F$. Clearly $\chi = \{\chi_{\alpha,\beta}\colon \beta\ \omega\ \alpha\}$ satisfies (N1)–(N3) and their duals, so there is a normal coextension of F by the rectangular biordered sets E_α ($\alpha \in F$), by Theorem 5.2.

Let B and B' be regular partial bands. B is called a *normal* coextension of B' if there is a strong homomorphism θ from B onto B' such that the following condition is satisfied: if $e \in B$, $e\theta = \alpha$ and $\beta\ \omega\ \alpha$ in B' then there is a unique $f \in B$ such that $f\ \omega\ e$. Clearly if B is a normal coextension of B' then $E(B)$ is a normal coextension of $E(B')$. We have the following method for constructing normal coextensions of regular partial bands.

Theorem 5.4. *Let B' be a regular partial band with $E(B') = F$ and let $E = E(F, \{E_\alpha\}, \chi)$ be a normal coextension of F by the rectangular biordered sets E_α ($\alpha \in F$). Let B be the richest regular partial band such that the natural bimorphism $\theta\colon E \to F$ extends to a homomorphism from B onto B'. Then B is a normal coextension of B' and every normal coextension of B' is obtained this way.*

Proof. By Theorem 4.8 we need only check that the ϕ and ψ mappings associated with the coextension E of F satisfy condition (P5). So let

$$\begin{bmatrix} e_0 & e_1 \\ e_3 & e_2 \end{bmatrix}$$

be an E-square in E, $\theta(e_i) = \alpha_i$ for $i = 0, 1, 2, 3$ and

$$\begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 \end{bmatrix}$$

a rectangular band in F, and let $t = (i, \lambda) \in \omega(e_0)$ with $\theta(t) = \beta$. Suppose that $\phi^{e_3}_{r(\beta)}(i) = j$ and $\phi^{e_2}_{r(\beta)}(i) = k$. By definition of the ϕ mappings associated with the normal coextension (proof of Theorem 5.3) we see that $e\chi_{\alpha_3,\alpha_3\beta\alpha_3} = (j, \mu)$ and $e_2\chi_{\alpha_2,\alpha_2(\beta\alpha_1)\alpha_2} = (k, \nu)$ for some μ, ν. But $e_2 \mathcal{R} e_3$ and $\alpha_3\beta\alpha_3 = \alpha_3\beta\mathcal{R}(\alpha_3\beta)\alpha_2 = \alpha_2(\beta\alpha_1) = \alpha_2(\beta\alpha_1)\alpha_2$ since

$$\begin{bmatrix} \alpha_0 & \alpha_1 \\ \alpha_3 & \alpha_2 \end{bmatrix}$$

is τ-commutative in F, so by condition (N2), $j = k$. Similarly $\lambda\psi^{e_1}_{l(\beta)} = \lambda\psi^{e_2}_{l(\beta)}$, so condition (P5) is satisfied as required.

We close the paper by reformulating Theorem 5.2 in category-theoretical language. Let $E = E(F, \{E_\alpha\}, \chi)$ be a normal coextension of a biordered set F. Let

$$I_F = \{r(\alpha): \alpha \in F\}, \qquad \Lambda_F = \{l(\alpha): \alpha \in F\}.$$

Then I_F and Λ_F are partially ordered sets with the obvious partial orders; viz: $r(\alpha) \leqslant r(\beta)$ iff $\alpha\omega^r\beta$ and $l(\alpha) \leqslant l(\beta)$ iff $\alpha\omega^l\beta$.

For each $r(\alpha) \in I_F$ let $\chi_R(r(\alpha)) = I_\alpha$ and for each $r(\alpha), r(\beta) \in I_F$ with $r(\alpha) \leqslant r(\beta)$ and $i \in I_\beta$ let $i\chi_R(r(\beta), r(\alpha)) = j$ where for some $\lambda \in \Lambda_\alpha$, $(i, \lambda)\chi_{\beta,\alpha\beta} = (j, \mu)$ for some $\mu \in \Lambda_{\alpha\beta}$. Then axiom (N2) implies that $\chi_R(r(\beta), r(\alpha))$ is a well-defined function from I_β into I_α. Also the hypothesis $I_\alpha = I_{\alpha'}$ if $\alpha\mathcal{R}\alpha'$ implies that $\chi_R(r(\alpha))$ is also well defined. Thus it follows from axioms (N1) and (N2) that χ_R is a functor from I_F into the category of sets. Dually, we have a functor χ_L: $\Lambda_F \to$ Sets. Note that for each $\alpha \in F$, $E_\alpha = \chi_R(r(\alpha)) \times \chi_L(l(\alpha))$ and for $\beta\ \omega\ \alpha$ and $e = (i, \lambda) \in E_\alpha$,

$$e\chi_{\alpha,\beta} = (i\chi_R(r(\alpha), r(\beta)), \lambda\chi_L(l(\alpha), l(\beta))).$$

Suppose conversely that we are given two functors χ_R: $I_F \to$ Sets and χ_L: $\Lambda_F \to$ Sets. If we define E_α $(\alpha \in F)$ and $\chi_{\alpha,\beta}$ $(\beta\ \omega\ \alpha)$ as above, it is immediate that the family of mappings $\chi = \{\chi_{\alpha,\beta}\}$ satisfies conditions (N1), (N2), (N3) and their duals. Thus we may reformulate Theorem 5.2 as follows.

THEOREM 5.5. *Let F be a biordered set and χ_R and χ_L be two set-value functors from the partially ordered sets $I_F = F/\mathcal{R}$ and $\Lambda_F = F/\mathcal{L}$ respectively. For each $\alpha \in F$, write*

$$E_\alpha = \chi_R(r(\alpha)) \times \chi_L(l(\alpha)) \quad \textit{and} \quad E = \bigcup_{\alpha\in F} E_\alpha.$$

For $(i, \lambda) \in E_\alpha$ and $(j, \mu) \in E_\beta$, define

$$(i, \lambda)(j, \mu) = \begin{cases} (j, \mu) & \textit{if } \beta\omega^r\alpha, j = i\chi_R(r(\alpha), r(\beta)), \\ (i, \mu\chi_L(l(\beta), l(\alpha\beta))) & \textit{if } \alpha\omega^r\beta, i = j\chi_R(r(\beta), r(\alpha)), \\ (i, \lambda) & \textit{if } \alpha\omega^l\beta, \lambda = \mu\chi_L(l(\beta), l(\alpha)), \\ (i\chi_R(r(\alpha), r(\alpha\beta)), \mu) & \textit{if } \beta\omega^l\alpha, \mu = \lambda\chi_L(l(\alpha), l(\beta)). \end{cases}$$

This defines a partial binary operation in E and relative to this E becomes a biordered set which is a normal coextension of F by rectangular biordered sets $\{E_\alpha\}$. Every normal coextension of F can be constructed in this way.

REMARK. If F is a semilattice, it is clear that we may identify I_F and Λ_F with F. Thus the pair of mappings $(\chi_R(\alpha, \beta), \chi_L(\alpha, \beta))$ (for $\beta\ \omega\ \alpha$) induces a homomorphism of E_α into E_β. Thus if we define $\chi(\alpha) = E_\alpha$ $(\alpha \in F)$, $\chi(\alpha, \beta) = (\chi_R(\alpha, \beta), \chi_L(\alpha, \beta))$ $(\beta\ \omega\ \alpha)$, then χ becomes a functor from the semilattice F to the category of rectangular bands. We thus obtain the classical representation of normal bands as functors from semilattices to the category of rectangular bands (or equivalently as "strong" semilattices of rectangular bands–cf. Yamada and Kimura [**31**] or Howie [**14**, Theorem 5.14]).

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